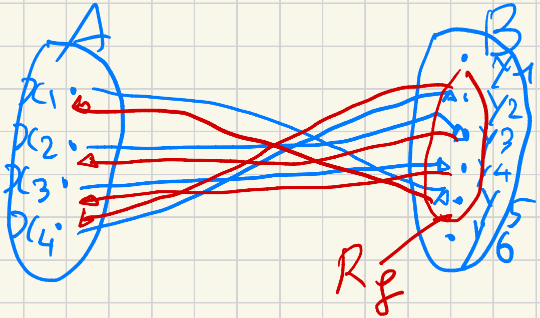
INVERSE FUNCTIONS

DEF Given $f : A \to B$, $R_f = \{ y \in B :$ there exists at least one $x \in A$ which satisfies $f(x) = y \}$

f is said to be 1-to-1 if for each x_1, x_2 in A with $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$

Given a 1-to-1 function f, it is possible to define its inverse function $f^{-1} : R_f \to A$, and for each $y \in R_f$, $f^{-1}(y) = x \in A$ such that $f(x) = y$.

$f(x_1) = y_5$
$f(x_2) = y_3$
$f(x_3) = y_4$
$f(x_4) = y_2$



$R_f = \{ y_2, y_3, y_4, y_5 \}$ f is 1-to-1

hence f^{-1} exists

$f^{-1} : R_f \to A$, $f^{-1}(y_2) = x_4$

$f^{-1}(y_3) = x_2$, $f^{-1}(y_4) = x_3$ $f^{-1}(y_5) = x_1$

$f^{-1}(y_1)$ is not defined, and no $f^{-1}(y_6)$

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x - 3$; f is 1-to-1, hence there exists f^{-1}; $R_f = \mathbb{R} = (-\infty, +\infty)$

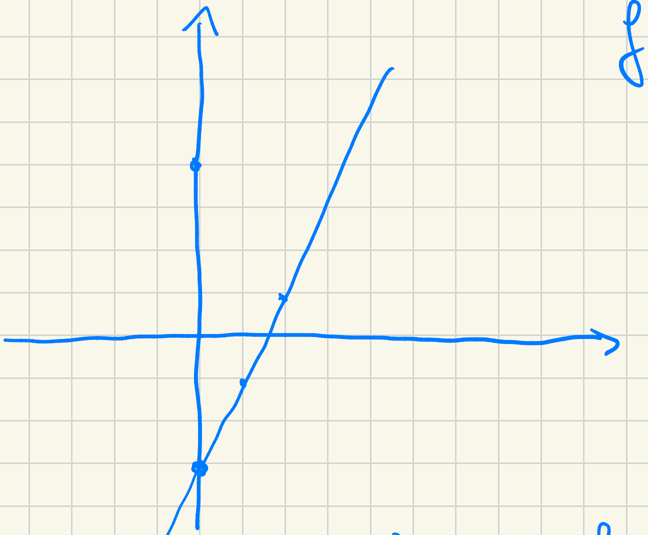
Given $y \in \mathbb{R}$, $\boxed{f^{-1}(y) = x} \in \mathbb{R}$:

$f(x) = y \iff 2x - 3 = y \iff$

unknown

$2x = y + 3 \iff x = \frac{1}{2}y + \frac{3}{2}$ hence

$f^{-1}(y) = \frac{1}{2}y + \frac{3}{2}$



$f(0) = -3$, $f(1) = -1$, $f(2) = 1$

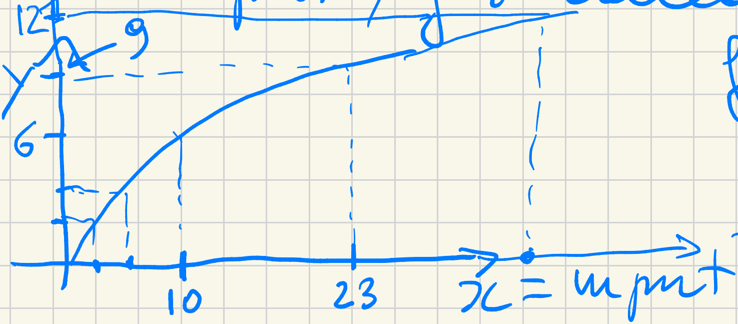
$f(x) = 2x-3$, $f^{-1}(y) = \frac{1}{2}y + \frac{3}{2}$

to obtain quantity y of output

$f^{-1}(5) = 4$ because $f(4) = 2 \cdot 4 - 3 = 5$

$f^{-1}(19) = \frac{19}{2} + \frac{3}{2} = \frac{22}{2} = 11$ because $f(11) = 22-3$
$= 19$

For a firm, f ~~a~~ associates to each non-negative quantity of input, a certain quantity of output, ~~opt~~ obtained using the input; f is called production function



f^{-1} associates to each quantity y of output the quantity $f^{-1}(y)$ of input necessary

Two properties of inverse functions:

$f^{-1}(f(x)) = x$ for each $x \in A$

$f(f^{-1}(y)) = y$ for each $y \in R_f$

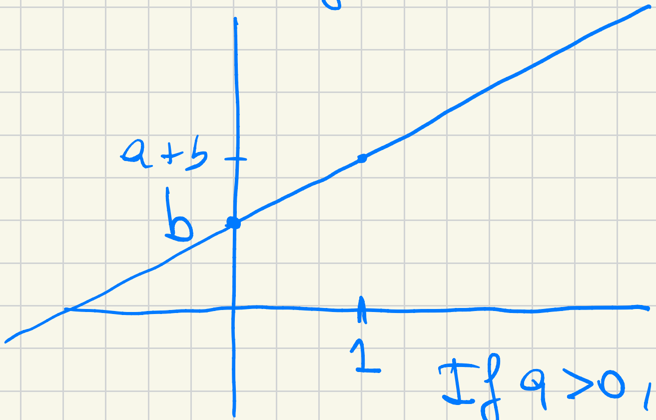
ELEMENTARY FUNCTIONS

LINEAR FUNCTION

A function f is said to be linear if $f : \mathbb{R} \to \mathbb{R}$ and

$$f(x) = \underset{\substack{\uparrow \\ \text{slope of } f}}{a}x + \underset{\substack{\uparrow \\ \text{intercept of } f}}{b}$$ for some a, b in $\mathbb{R}$

$f(x) = 2x - 3$ is a linear function with $a = 2$, $b = -3$



$f(0) = b$

$f(1) = a + b$

decreases

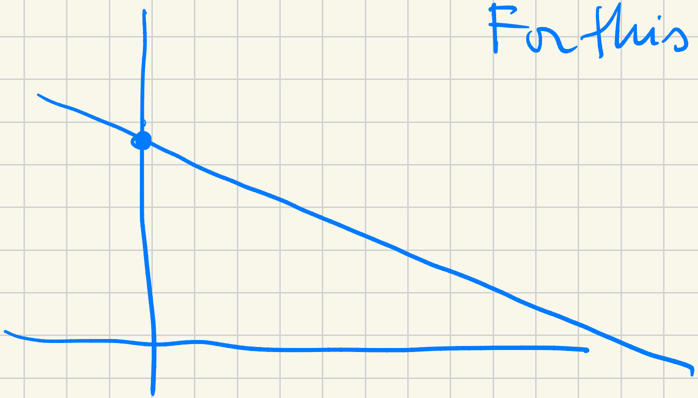
If b increases, then the graph of f moves up in a parallel way

down

If $a > 0$, then f is a monotone strictly increasing function

If $a < 0$ " " " " " decreasing function

If $a = 0$, " f is constant: $f(x) = b$ for each $x \in \mathbb{R}$



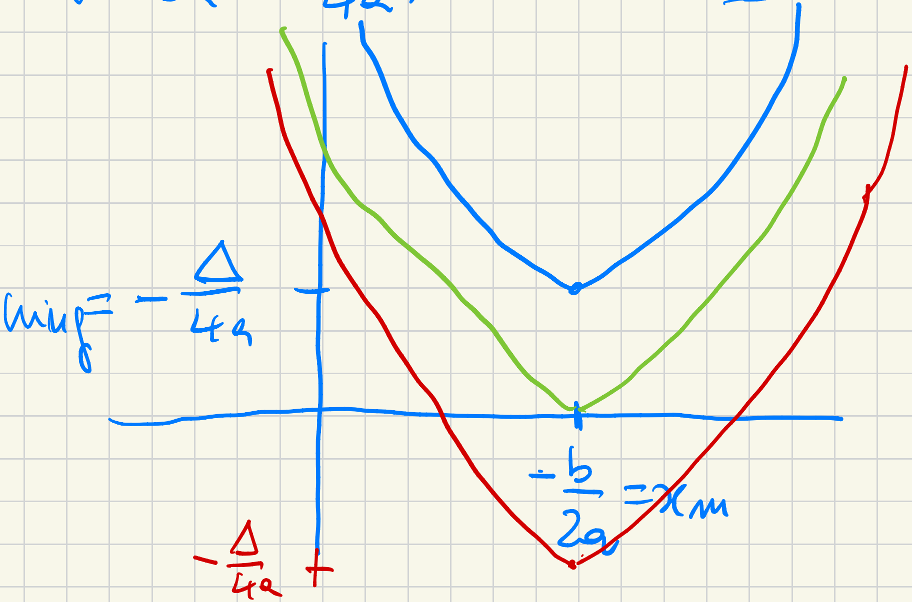
For this line, $b > 0$, $a < 0$

QUADRATIC FUNCTIONS

A function f is said to be quadratic if $f : \mathbb{R} \to \mathbb{R}$ and $f(x) = ax^2 + \underline{bx} + c$ for some a, b, c in $\mathbb{R}$, with $a \neq 0$

Let $\Delta = b^2 - 4ac$ and then $f(x)$ can be written as $a\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a}$

Case of $a>0$. When $a>0$, there exists a minimum point for f, which is $x_m = -\frac{b}{2a}$, and $\min f = -\frac{\Delta}{4a}$. The point with coordinates $\left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right)$ is called vertex of the graph of f



Say $\Delta<0$; then $-\frac{\Delta}{4a} > 0$

Say $\Delta>0$, then $-\frac{\Delta}{4a} < 0$

Say $\Delta=0$, then $-\frac{\Delta}{4a} = 0$

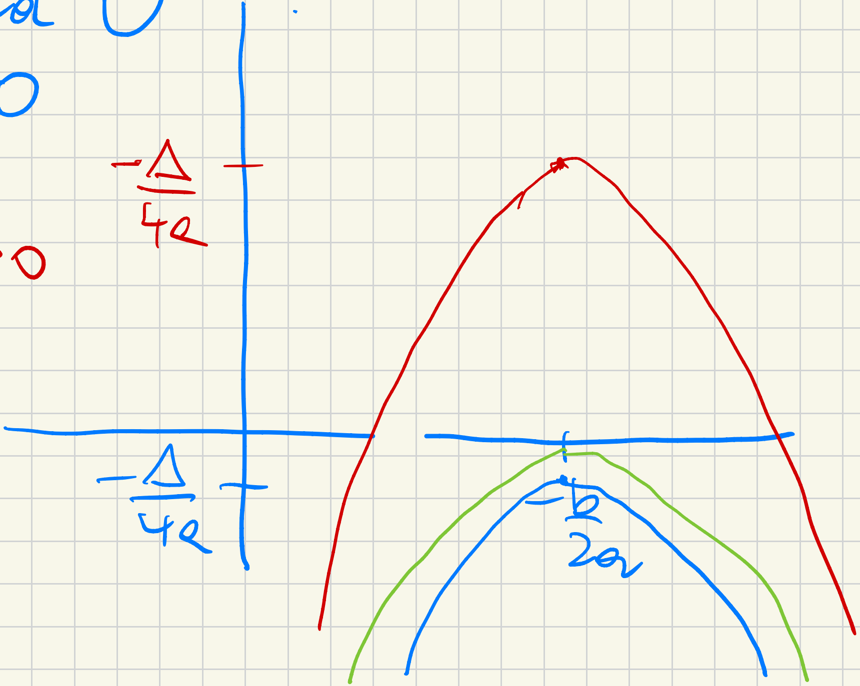
Case of a<0. When $a<0$, there exists a maximum point for f, which is $x_M = -\frac{b}{2a}$ and $\max f = -\frac{\Delta}{4a}$. The graph of f has the shape of a inverted U

Say $\Delta < 0$; then $-\frac{\Delta}{4a} < 0$

Say $\Delta > 0$; then $\underset{\max f}{\underset{\shortparallel}{-\frac{\Delta}{4a}}} > 0$

Say $\Delta = 0$; then

$-\frac{\Delta}{4a} = 0$



$f(x) = 2x^2 - \frac{3}{5}x + \frac{9}{10}$, $g(x) = -\frac{1}{3}x^2 + 5x - 11$

POLYNOMIALS

f is said to be a polynomial function if

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ for some n positive integer and $a_n, a_{n-1}, \ldots, a_1, a_0$ in $\mathbb{R}$

For instance,

$f(x) = 5x^3 + 7x^2 - 13x + 6$ is a polynomial function with

$n = 3$, $a_3 = 5$, $a_2 = 7$, $a_1 = -13$, $a_0 = 6$

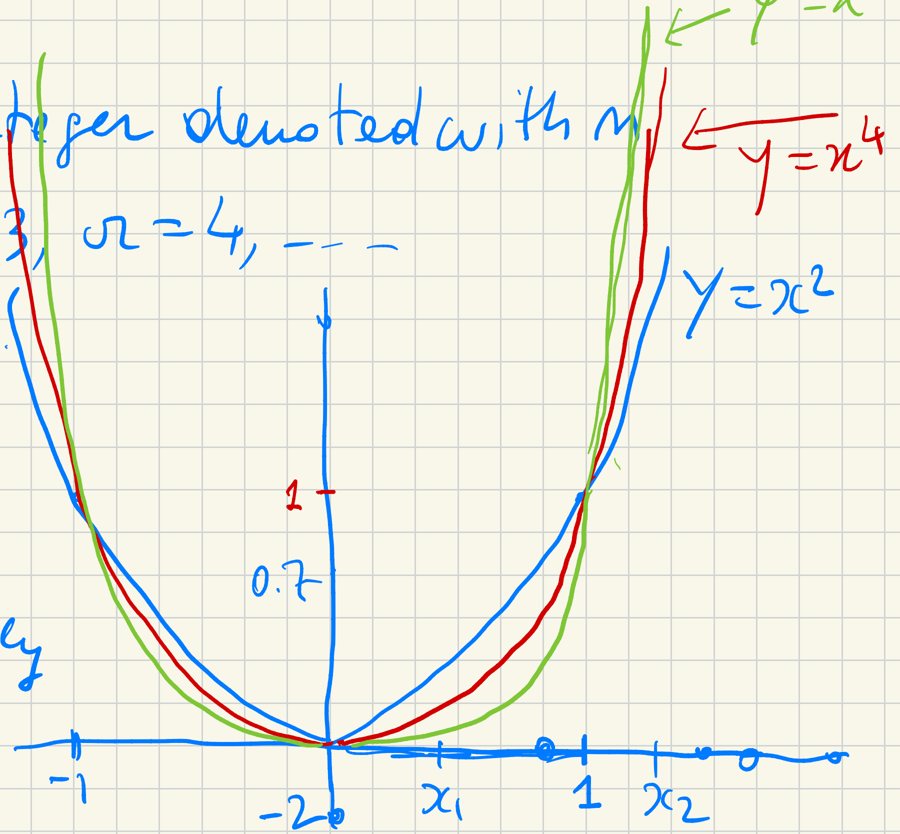
POWER FUNCTION

f is said to be a power function if $f(x) = x^a$ with $a \in \mathbb{R}$.

Case of a = a positive integer denoted with n

Here $a = 1$, $n = 2$, $n = 3$, $n = 4$, ---

First consider a even



$y = x^4$

$y = x^6$

$f(x) = x^4$ is monotone strictly decreasing in $(-\infty, 0]$

f is monotone strictly increasing in $[0, +\infty)$

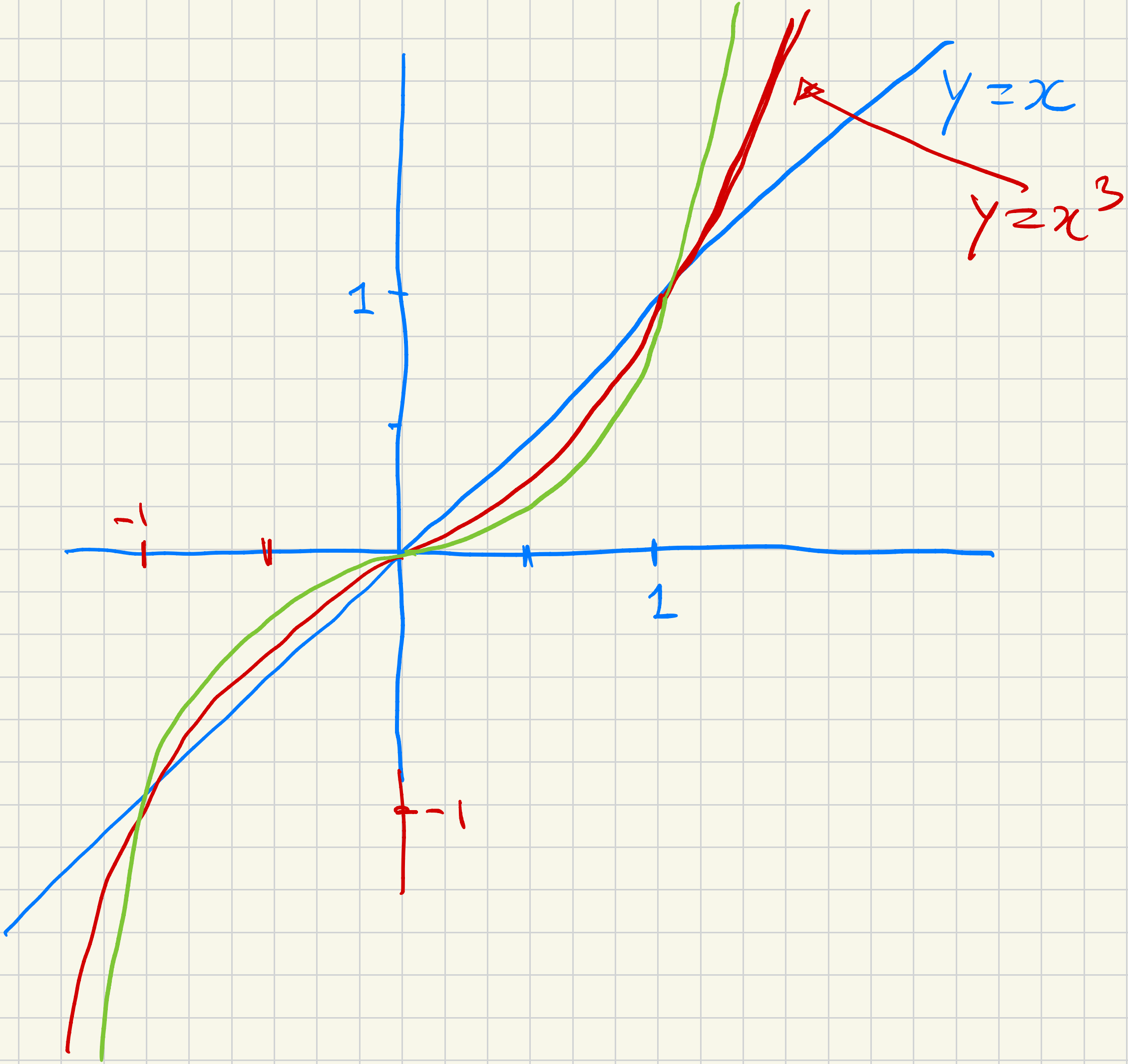
x_M does not exist

$x_m = 0$, $\min f = 0$

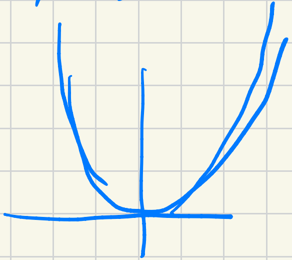
Now a odd

$y = x^3$

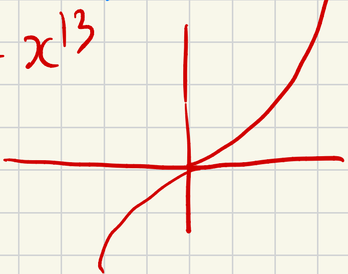
$y = x^5$



$f(x) = x^{14}$

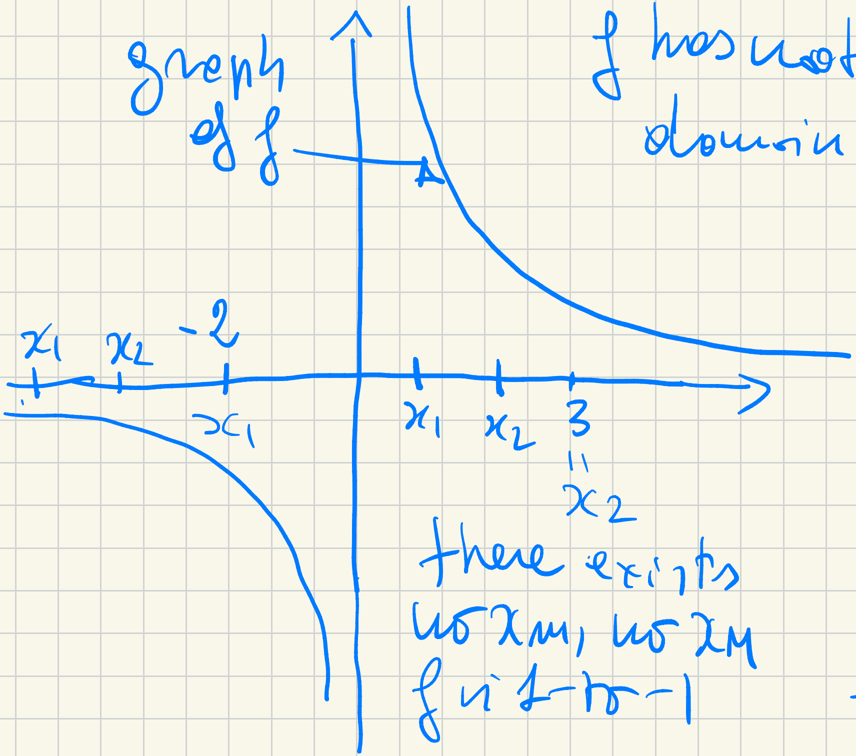


$g(x) = x^{13}$



Case of $a=-1$, $a=-2$, that is $f(x)=x^{-1}$, $g(x)=x^{-2}$

$= \frac{1}{x}$ $\qquad$ $= \frac{1}{x^2}$

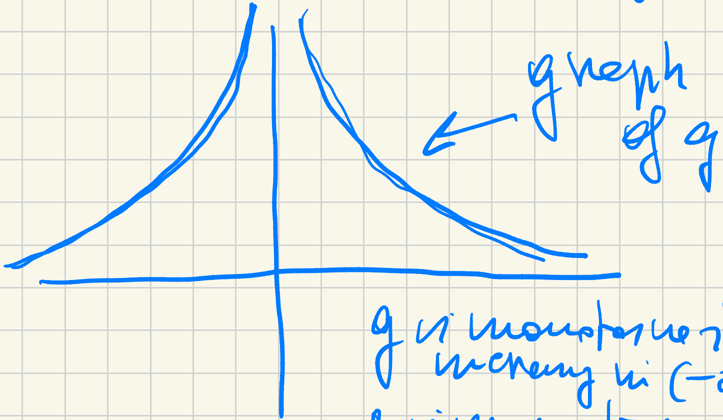


f has natural domain $= \underbrace{(-\infty,0)}_{\text{here } f \text{ is strictly decreasing}} \cup \underbrace{(0,+\infty)}_{\text{here } f \text{ is strictly decreasing}} = R_f$

there exists no x_m, no x_M

f is 1-to-1

$f(\underset{\overset{\shortparallel}{x_1}}{-2}) = -\frac{1}{2}$, $f(\underset{\overset{\shortparallel}{x_2}}{3}) = \frac{1}{3}$



g is monotone strictly increasing in $(-\infty,0)$

g is monotone strictly decreasing in the interval $(0,+\infty)$;

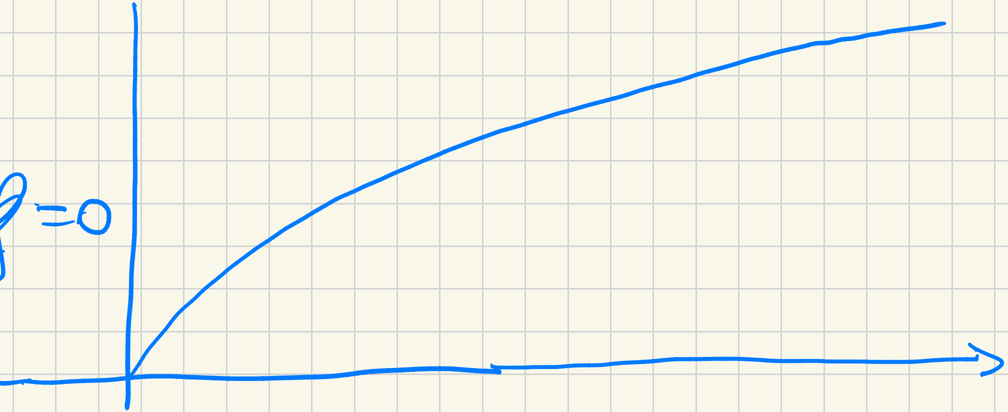
Case of $a = 1/2$; then $f(x) = x^{1/2} = \sqrt{x}$, has natural domain $[0, +\infty)$

Given $x \geq 0$, $\sqrt{x}$ is the unique non-negative number such that its square is equal to x.

$\sqrt{25} = 5$.

$x_m = 0$, $\min f = 0$

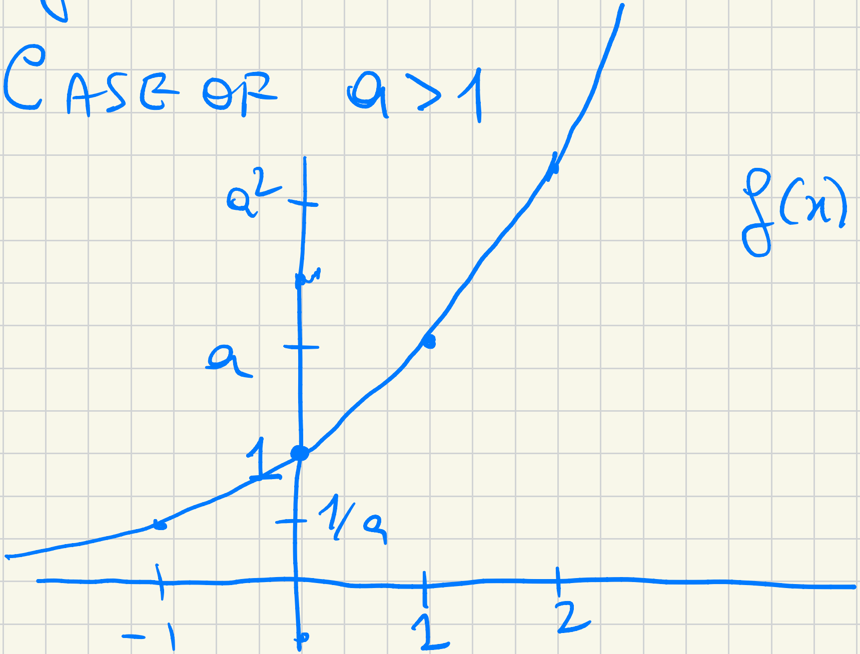
x_M does not exist



EXPONENTIAL FUNCTIONS

A function is said to be exponential if $f: \mathbb{R} \to \mathbb{R}$, $f(x) = a^x$ with $a > 0$, $a \neq 1$, this is different from x^a

CASE OF $a > 1$



$f(x) = a^x$, $f(0) = 1$ $R_f = (0, +\infty)$

$f(1) = a$

$f(2) = a^2$

$f(-1) = \frac{1}{a}$

CASE OF $a \in (0,1)$. Then $f(0) = 1$

$0<a<1$

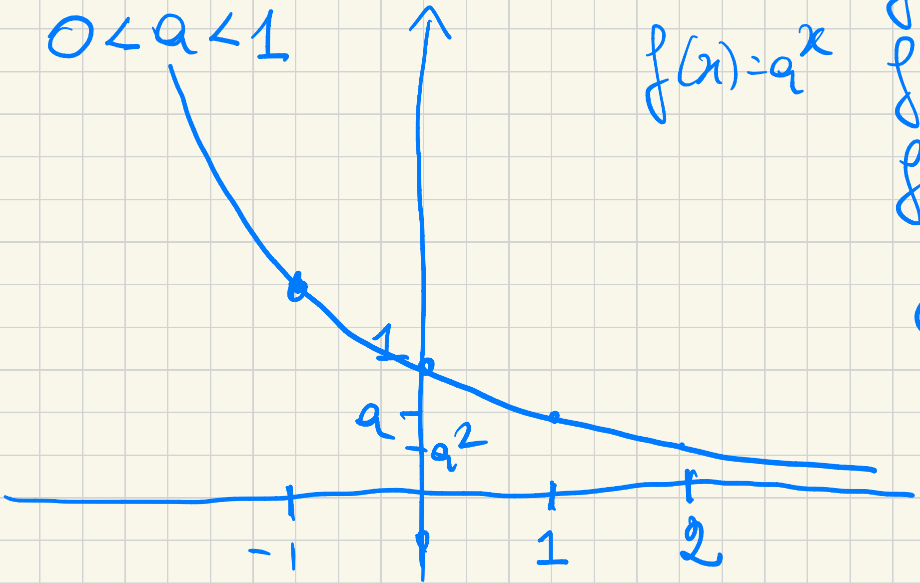
$f(x)=a^x$

$f(1) = a$

$f(2) = a^2$

$f(-1) = \frac{1}{a}$

$R_f = (0, +\infty)$



log of x in base a

DEF Given $x>0$ and $a>0$, $a \neq 1$, $\log_a x$ is the exponent to give to a in order to obtain x as a result.

$\log_5 25 = 2$ because $5^2 = 25$ $\log_4 64 = 3$ as $4^3 = 64$

$\log_{64} 8 = 1/2$ $\log_6 1 = 0$ as $6^0 = 1$

f is said to be a log function if $f(x) = \log_a x$ for some $a>0$, $a \neq 1$

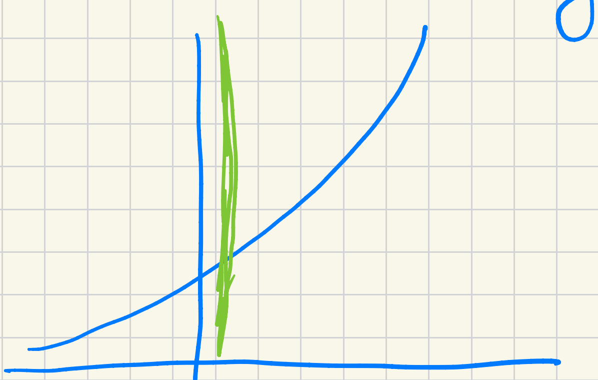
CASE OF $a > 1$. $g(x) = a^x$, $g: \mathbb{R} \to \mathbb{R}$; g is 1-to-1

hence there exists g^{-1}

$R_g = (0, +\infty)$, hence $g^{-1}: (0,+\infty) \to \mathbb{R}$

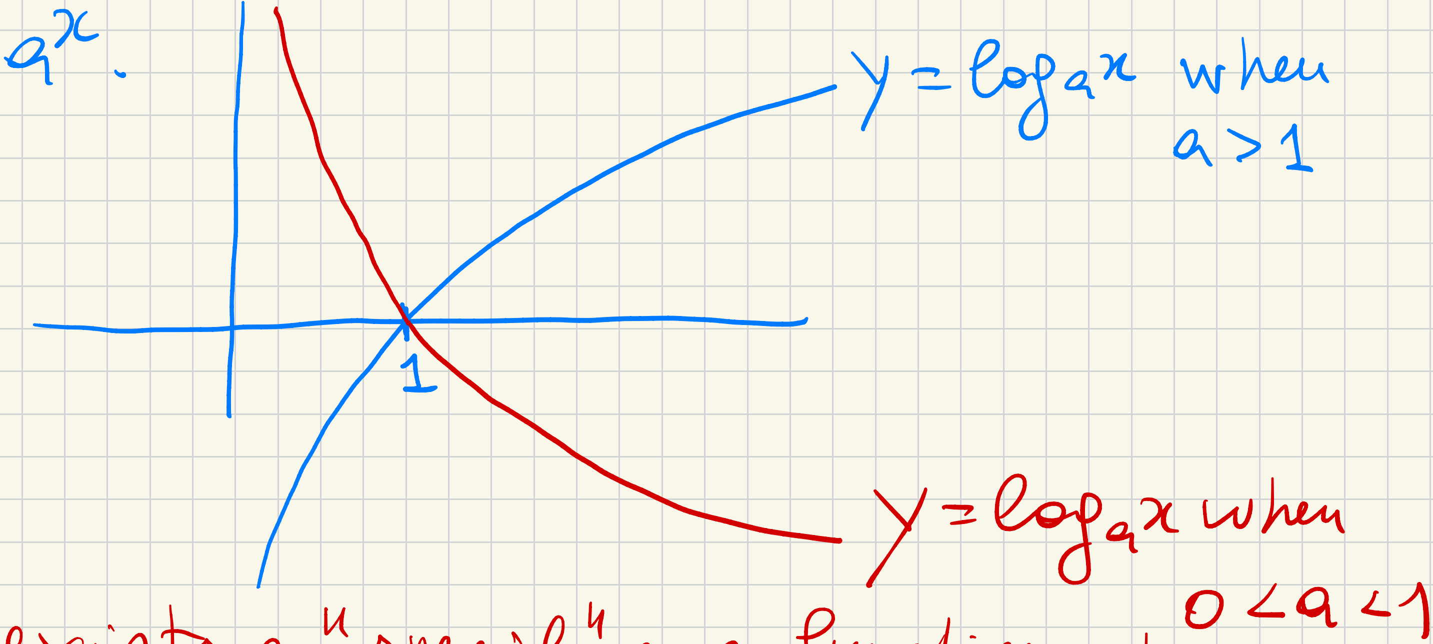
Given $y \in (0,+\infty)$, $g^{-1}(y) =$

$x \in \mathbb{R}$ such that $g(x) = y \iff$



this has solution $\log_a y$, hence $g^{-1}(y) = \log_a y$ $\longleftarrow$ $a^x = y$; I look for the exponent to give to a to obtain y, that is $\log_a y$

Hence $f(x) = \log_a x$ is the inverse function of $g(x) = a^x$.



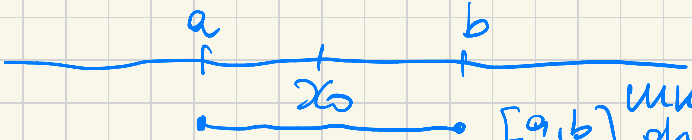
There exists a "special" exp. function which is $f(x) = \underline{e}^x$, with $e \cong 2.71$, and a "special" log function $g(x) = \log_e x = \ln x$

LIMITS AND CONTINUITY

Given $f: A \to \mathbb{R}$ (with $A \subseteq \mathbb{R}$), I want to give a meaning to $\lim_{x \to x_0} f(x) = L$, in which L is a real number

$\lim_{x \to x_0} f(x)$ is about $f(x)$ when x is close to x_0 but $x \neq x_0$

In order to make sense to ask the value of $\lim_{x \to x_0} f(x)$, x_0 needs to satisfy the following condition: For each a, b such that $a < x_0 < b$, the interval $[a, b]$ needs to include point of A (the domain of f) different from x_0

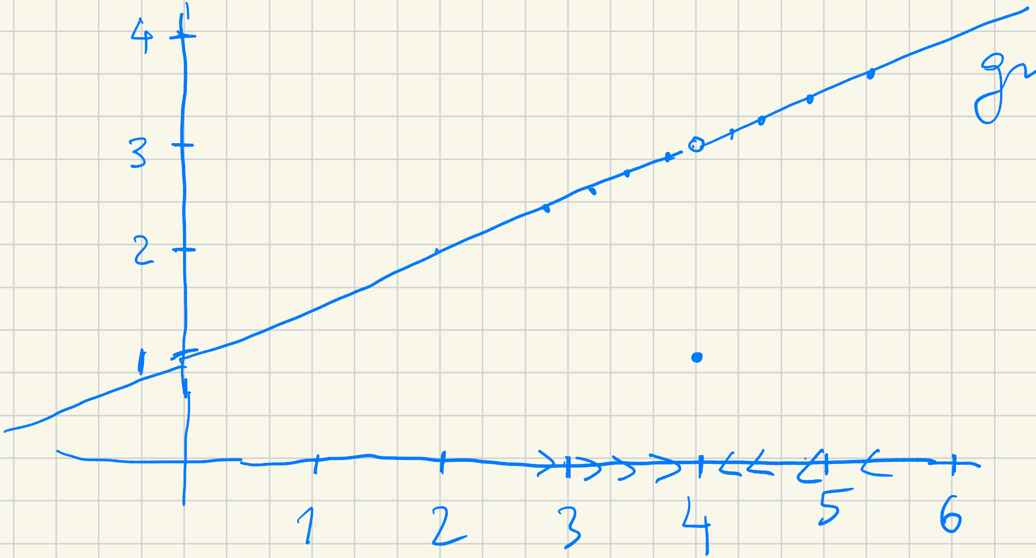


$[a,b]$ must include points of A different from x_0, even though a is very close to x_0 and b is very close to x_0

The previous condition implies that $f(x)$ can be evaluated when x is close to x_0, hence it makes sense to try to evaluate $\lim_{x \to x_0} f(x)$. If the condition is violated, then $\lim_{x \to x_0} f(x)$ has no meaning.

The writing $\lim_{x \to x_0} f(x) = L$ is (loosely) interpreted as follows: when x is close to x_0 (or x approaches x_0, written as $x \to x_0$), we have that $f(x)$ is close to x_0 (or $f(x)$ tends to L).

Example: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 1 + \frac{1}{2}x & \text{if } x \neq 4 \\ 1 & \text{if } x = 4 \end{cases}$



$\lim_{x \to 4} f(x)$ is about $f(x)$ when x is close to 4, but $x \neq 4$. The graph suggests that as $x \to 4$, $f(x)$ tends to 3, thus $\lim_{x \to 4} f(x) = 3$.

This result does not depend on $f(4)$.