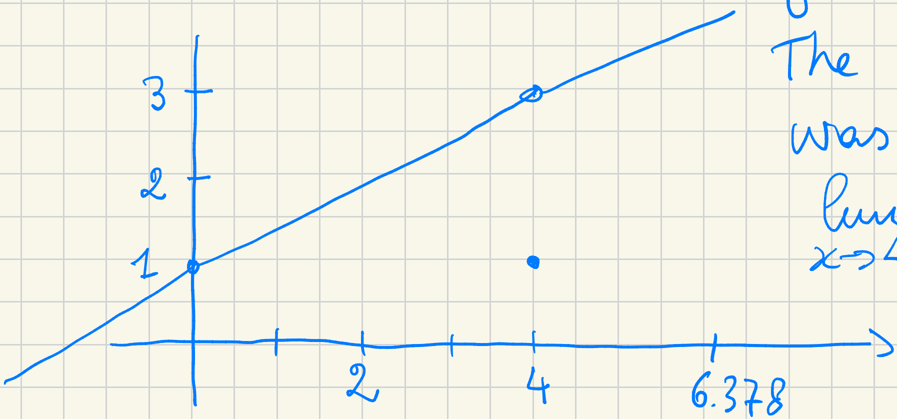
LIMITS AND CONTINUITY

DEF Given $f: A \to \mathbb{R}$, with $A \subseteq \mathbb{R}$ and $x_0 \in A$, f is said to be continuous at x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

Hence when x is close to x_0, then $f(x)$ is close to $f(x_0)$



The f from last class was such that

$$\lim_{x \to 4} f(x) = 3 \text{ but } f(4) = 1$$

This f is not continuous at $x_0 = 4$

DEF Given $f: A \to \mathbb{R}$ (with $A \subseteq \mathbb{R}$), f is said to be continuous in A if it is continuous at each point of A.

Every elementary function is continuous in its natural domain. For instance, $f(x) = \frac{1}{x}$, $f: \underbrace{(-\infty, 0) \cup (0, +\infty)}_{\mathbb{R} - \{0\}} \to \mathbb{R}$

f is continuous at $x_0 = 3$, that is

$$\boxed{\lim_{x \to 3} f(x)} = f(3) = \boxed{\frac{1}{3}}$$

$g(x) = 2^x$, $g: \mathbb{R} \to \mathbb{R}$, $\boxed{\lim_{x \to 5} g(x)} = g(5) = 2^5 \boxed{= 32}$

If f and g are both continuous at x_0, then

$f + g$ is continuous at x_0: the sum of continuous functions is continuous

$f - g$ is continuous at x_0

$f \cdot g$ " " "

f / g " " " provided that $g(x_0) \neq 0$

$h(x) = \underbrace{3^x}_{f(x)} \cdot (\underbrace{1 + 5x - 7x^2}_{g(x)})$ is continuous as it is the product of continuous functions

$f(x)$ is continuous as it is an elementary function

$g(x)$ is continuous as it is the sum of continuous functions

$$\boxed{\lim_{x \to 2} h(x)} = h(2) = 3^2(1 + 10 - 7 \cdot 4) = \boxed{\ldots}$$

If g is continuous at x_0 and f is continuous at $g(x_0)$, then the $f \circ g$ is continuous at x_0.

$g: \mathbb{R} \to \mathbb{R}$, $g(x) = 3 + 8x^2$ is continuous in $\mathbb{R}$

$f: (0, +\infty) \to \mathbb{R}$, $f(x) = \ln x$ " " in $(0, +\infty)$

$h(x) = f(g(x)) = \ln(3 + 8x^2)$ is continuous in $\mathbb{R}$, as it is the composition of continuous functions

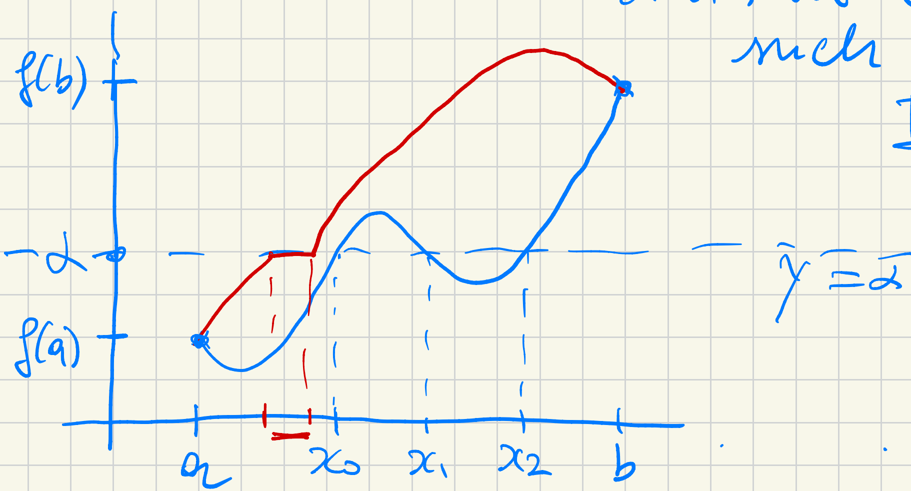
$$\boxed{\lim_{x \to 3} h(x)} = h(3) = \ln(3 + 72) \boxed{= \ln 75}$$

Intermediate Value Theorem

H) Suppose that $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$

P) and $f(a) \neq f(b)$.

Thesis: For each α between $f(a)$ and $f(b)$, there exists at least one $x_0 \in (a,b)$ such that $f(x_0) = \alpha$



If f is strictly monotone, then there exists a unique $x_0 \in (a,b)$ such that $f(x_0) = \alpha$

Special case of the IVT

Suppose that $f:[a,b]\to\mathbb{R}$ is continuous and $f(a)<0$, $f(b)>0$, or $f(a)>0, f(b)<0$. Hp

There exists at least one $x_0\in(a,b)$ such that $f(x_0)=0$ (this is the case when $\alpha=0$ in the IVT) and x_0 is unique if f is strictly monotone. thesis

Example $2^x+7\cdot\sqrt{x+1}=18$. This is equivalent to $\underbrace{2^x+7\sqrt{x+1}-18}_{f(x)}=0$. The equation is written as $f(x)=0$, with f a continuous function

$f(-1)=2^{-1}+7\sqrt{-1+1}-18=0.5+0-18=-17.5$

$f(0)=1+7-18=-10$, $f(2)\simeq-2$, $\boxed{f(3)=8+14-18=4}$

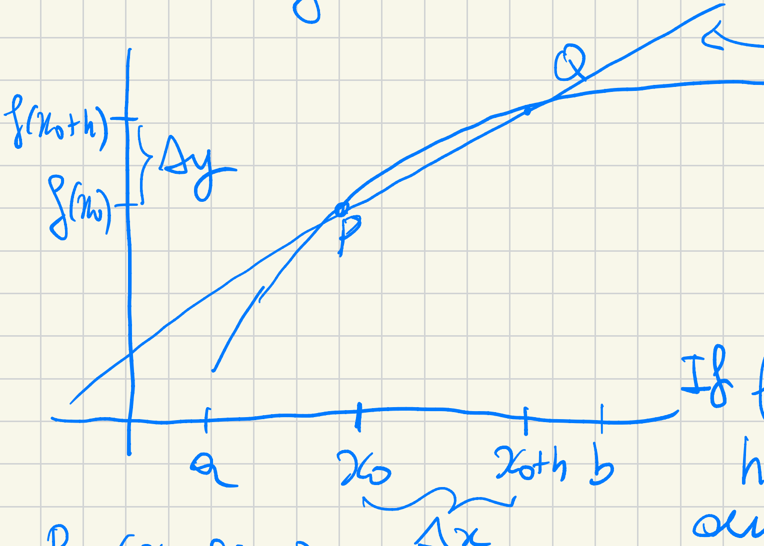
I apply the IVT with $a=2$, $b=3$ as $f(2)<0$, $f(3)>0$. Hence there exists at least one $x_0 \in (2,3)$ s.t. $f(x_0)=0$

I could have taken $a=0$, $b=3$, and then the theorem would reveal that there exists at least one $x_0 \in (0,3)$ s.t. $f(x_0)=0$, a less precise claim than the previous one

f is strictly monotone increasing, hence there exists a unique solution for $f(x)=0$ in the interval $(2,3)$.

DIFFERENTIAL CALCULUS

Consider $f:(a,b)\to\mathbb{R}$ and $x_0\in(a,b)$



← line through P and Q, its slope is $\underbrace{\dfrac{f(x_0+h)-f(x_0)}{h}}_{\text{incremental quotient, a function of } h} = \dfrac{\Delta f}{\Delta x}$

$P=(x_0, f(x_0))$

$Q=(x_0+h, f(x_0+h))$

If $\lim\limits_{h\to 0} \dfrac{f(x_0+h)-f(x_0)}{h}$ exists and is finite, then its value is called derivative of f at x_0, denoted $f'(x_0)$

$f(x)=x^2-6x+5$, $x_0=4$. I want to evaluate

$$\lim_{h\to 0}\frac{f(4+h)-f(4)}{h}$$ in order to compute $f'(4)$

$$\frac{(4+h)^2-6(4+h)+5-(-3)}{h}=\frac{\cancel{16}+h^2+8h\cancel{-24}-6h\cancel{+5}\cancel{+3}}{h}=$$

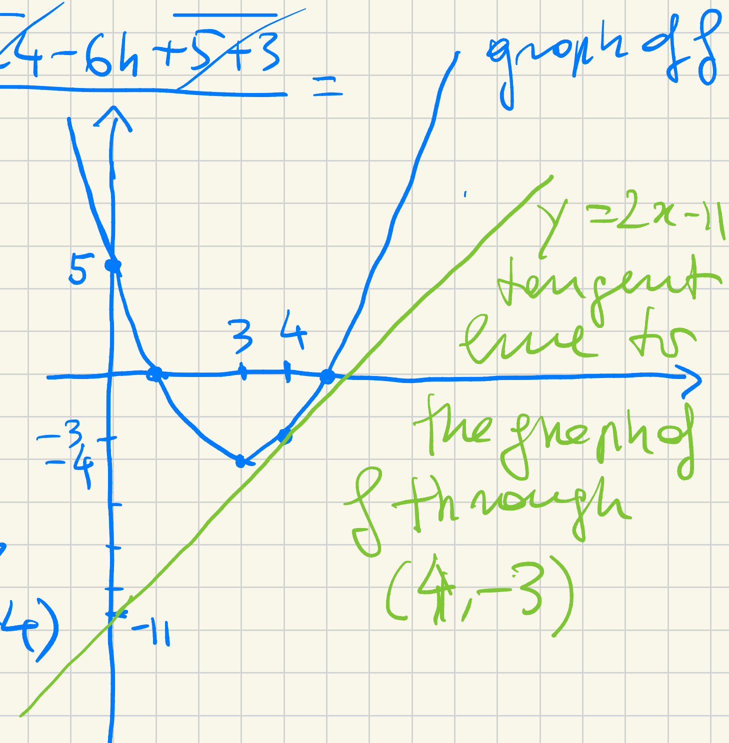
$$=\frac{h^2+2h}{h}=\frac{h^2}{h}+\frac{2h}{h}=h+2$$

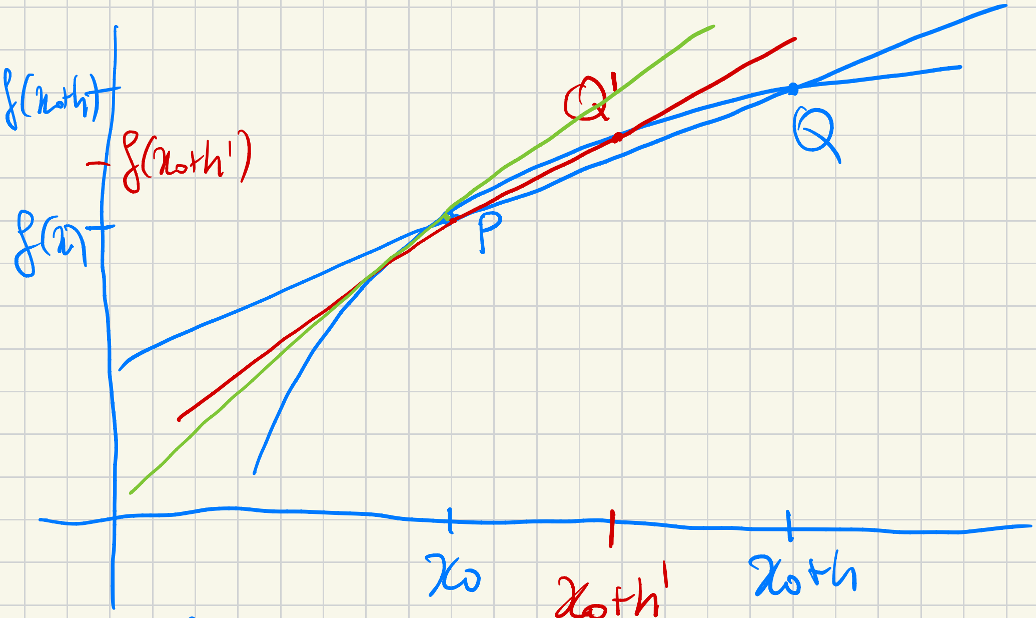
$$\lim_{h\to 0}(h+2)=2 \text{ hence } f'(4)=2$$

$f(4)=-3$. The tangent line to the graph of f at $(4,-3)$ has

$f'(4)=2$ equation $y=-3+2(x-4)$

$$y=2x-11$$





Take $0 < h' < h$. Then $\dfrac{f(x_0+h') - f(x_0)}{h'}$ is the slope of the red line

When h is close to 0, Q is close to P and the line through P and Q is close to the line in green, which has slope $\lim_{h \to 0} \dfrac{f(x_0+h) - f(x_0)}{h} = f'(x_0)$

The ~~line~~ green line is called tangent line to the graph of f, through $P = (x_0, f(x_0))$ and its equation is

$$y = f(x_0) + f'(x_0)(x - x_0)$$

If $f'(x_0)$ exists, then f is said to be differentiable at x_0.

If f is differentiable at each $x \in (a,b)$, then f is said to be differentiable in (a,b)

F.i., $f(x)=x^2-6x+5$ is differentiable at $x_0=4$, but in fact f is differentiable in $\mathbb{R}$

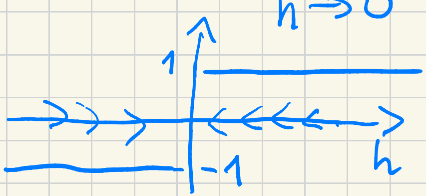
There exist functions which are not differentiable at some points, for example $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x<0 \end{cases}$ is such that $f'(0)$ does not exist; $\lim\limits_{h\to 0} \underbrace{\frac{f(0+h)-f(0)}{h}}$ does not exist

$|4|=4$

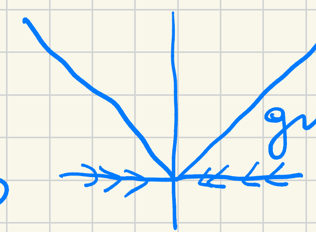
$|13|=13$

$|-6|=6$

$|-117|=117$



$\frac{f(h)}{h} = \frac{|h|}{h} = \begin{cases} 1 & \text{if } h>0 \\ -1 & \text{if } h<0 \end{cases}$



graph of f

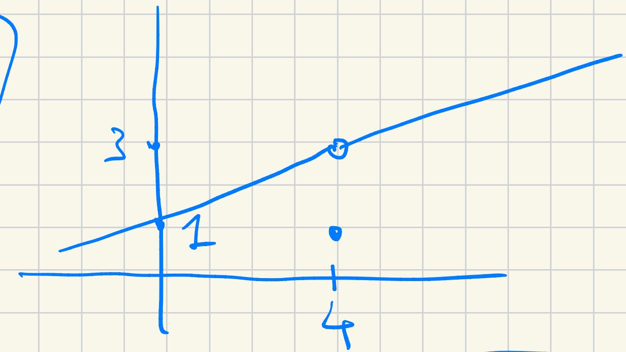
$|0|=0$

If f is differentiable at x_0, then it is continuous at x_0.

$$\boxed{f \text{ is differentiable at } x_0 \implies f \text{ is continuous at } x_0}$$

$$\not\Longleftarrow$$

because $f(x)=|x|$ is continuous at $x_0=0$, but it is <u>not</u> differentiable at $x_0=0$



$\boxed{\text{Is } f \text{ continuous at } x_0=4\text{? No}}$

Is f differentiable at $x_0=4$? No, because f is not continuous at $x_0=4$.

Notation. Sometimes, instead of writing f', a derivative is denoted with $\frac{df}{dx}$, y', $\frac{dy}{dx}$

COMPUTING DERIVATIVES

Derivatives for elementary functions:

$f(x)$	k	x	x^a	a^x	e^x	$\log_a x$	$\ln x$
$f'(x)$	0	1	$a x^{a-1}$	$a^x \cdot \ln a$	e^x	$\frac{1}{x \cdot \ln a}$	$\frac{1}{x}$

except for the case of $a \in (0,1)$ and $x = 0$

If f and g are differentiable, then

$h = f \pm g$ is differentiable and $h'(x) = f'(x) \pm g'(x)$

$h = af + k$ is differentiable and $h'(x) = a \cdot f'(x) + 0$

$h_1(x) = 3x^2 + 5$, $h_1'(x) = 3 \cdot 2x^1 + 0 = 6x$, $h_2(x) = 6 - 2\ln x$

$h_2'(x) = 0 - 2 \cdot \frac{1}{x} = -\frac{2}{x}$

$h_3(x) = 7e^x - 3$, $h_3'(x) = 7e^x$

$\sqrt{x} = x^{1/2}$

$h_4(x) = -2x^5 + \frac{4}{9}x + \frac{1}{15}\log_2 x + \frac{9}{16}3^x$

$\frac{7}{x} = 7 \cdot \frac{1}{x}$
$= 7 \cdot x^{-1}$

$h_4'(x) = -2 \cdot 5x^4 + \frac{4}{9} \cdot 1 + \frac{1}{15} \cdot \frac{1}{x \cdot \ln 2} + \frac{9}{16} \cdot 3^x \cdot \ln 3$

$h_5(x) = -3\sqrt{x} + \frac{7}{x} - 2\ln x + 11e^x + 9$

$h_5'(x) = -3 \cdot \frac{1}{2} \cdot x^{-\frac{1}{2}} + 7 \cdot (-1) \cdot x^{-2} - 2 \cdot \frac{1}{x} + 11e^x$

$= -\frac{3}{2\sqrt{x}} - \frac{7}{x^2} - \frac{2}{x} + 11e^x$

If f and g are differentiable, then $h = f \cdot g$ is differentiable and $h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

$$h(x) = \underbrace{x^2}_{f(x)} \cdot \underbrace{\ln x}_{g(x)} \qquad h'(x) = \underbrace{2x}_{f'} \cdot \underbrace{\ln x}_{g} + \underbrace{x^2}_{f} \cdot \underbrace{\frac{1}{x}}_{g'} =$$

$$= 2x \cdot \ln x + x = x(2\ln x + 1)$$