INTRODUCTORY COURSE FOR MATH, MASTER ECONOMICS & DEVELOPMENT

COURSE OUTLINE

Real valued functions of a real variable

Graph, Domain, Range, Inverse function

Monotonicity, Max/min points

Elementary functions: Linear, Quadratic, Power, Exp, Log functions

Limits & Continuity

Differential Calculus: Definition, Differentiation rules, linear approximation, Optimization, monotonicity test, $1^{st}/2^{nd}$ order conditions
Convex/concave functions

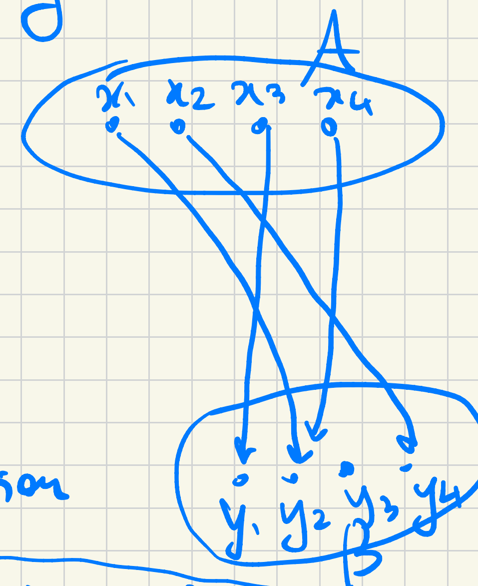
COURSE SCHEDULE

Mon., Sept. 9, 1.30-4pm, room D6/015
Tue, Sept. 10, 1.30-4pm, " D5/114
Thur, Sept. 12, 10 am-12.30pm, room D5/001
Fri, Sept. 13, 1.30-4pm, room D5/001

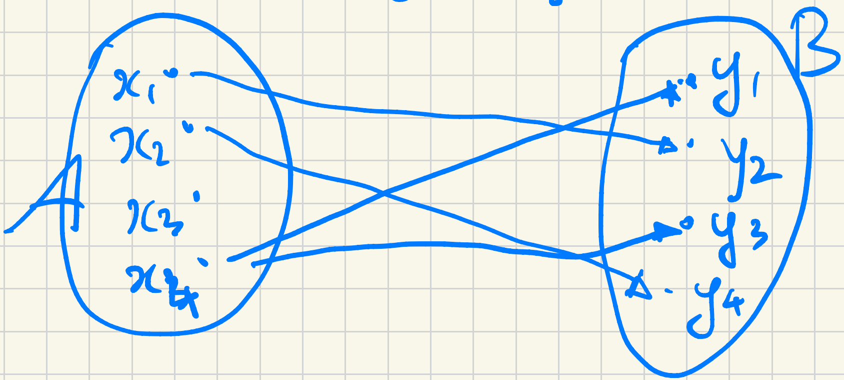
Many economic facts can be described through a function of one variable

- Production cost a firm incurs as a function of the quantity it produces
- Sales of a firm as a function of the price it charges for its product
- Consumption by a household as a function of the household's total income
- Interest accruing on a loan as a function of time

DEF Given two sets A and B, a function f from A to B is denoted as $f: A \to B$, and is a rule which to each $x \in A$ associates a $y \in B$, denoted $f(x)$



A = domain of the function
B = codomain of the function



not a function

$f(x_1) = y_1$, $f(x_2) = y_4$
$f(x_3) = y_1$, $f(x_4) = y_2$

I am interested in functions s.t. $A \subseteq \mathbb{R}$, $B = \mathbb{R}$: These functions are said real valued functions of a real variable

$f(x) \in \mathbb{R}$ for each x $\quad$ $x \in \mathbb{R}$

$x \in A$ is called independent variable

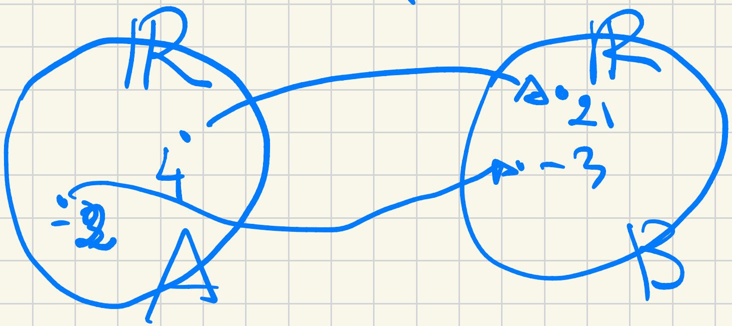
$y \in B$ is called dependent variable

$f(1) = 0$

Ex $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 + 2x - 3$

$f(4) = 4^2 + 2 \cdot 4 - 3 = 16 + 8 - 3 = 21$

f associates to $x = 4$ the real number 21



$f(-2) = 4 - 4 - 3 = -3$

Given $f: A \to \mathbb{R}$ (with $A \subseteq \mathbb{R}$), its graph is

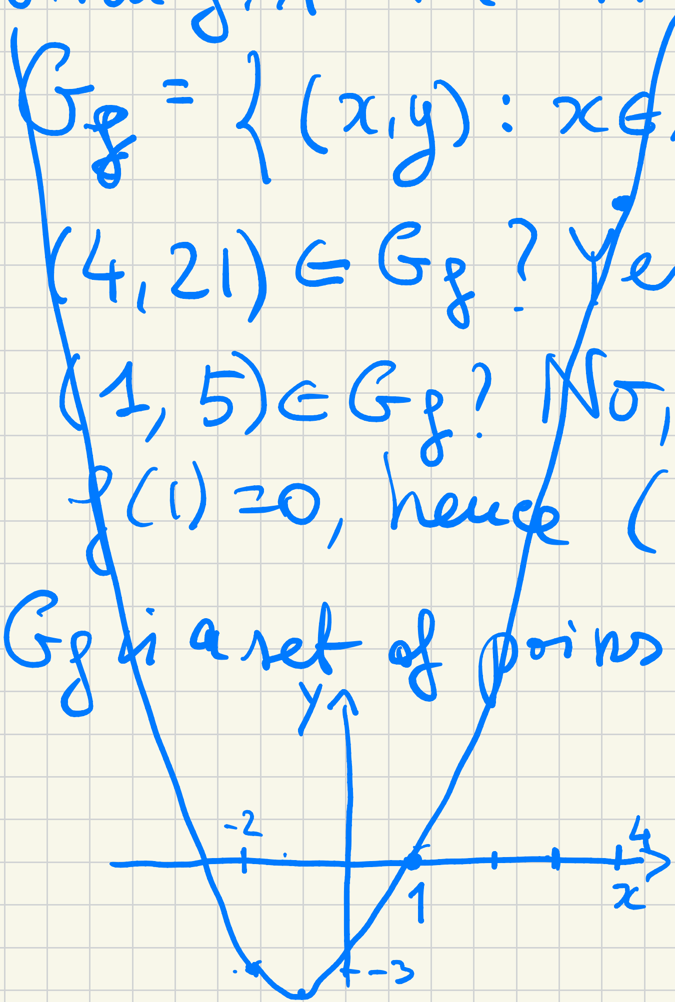
$$G_f = \{(x,y) : x \in A,\ y = f(x)\}$$

$(4,21) \in G_f$? Yes; $(-2,-3) \in G_f$

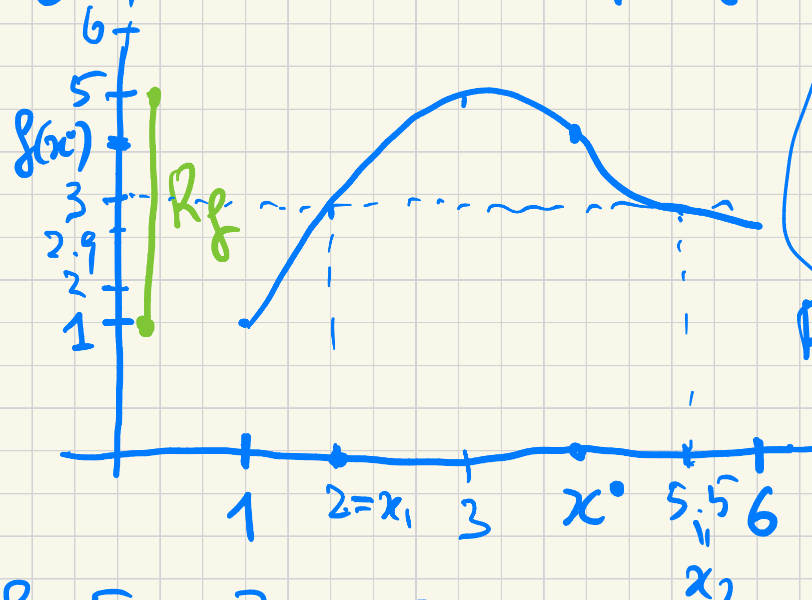
$(1,5) \in G_f$? No, because 5 is not equal to $f(1)$

$f(1) = 0$, hence $(1,0) \in G_f$

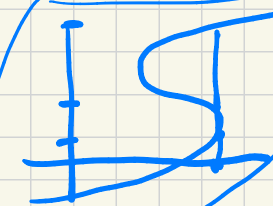
G_f is a set of points of real numbers



It is possible to represent a function only through its graph, without specifying its expression



this is not the graph of a function



$[1,6]$ = interval of real numbers: all real numbers between 1 and 6, including 1, including 6

1.274, 1.74532

1.8901732156

1.93333...

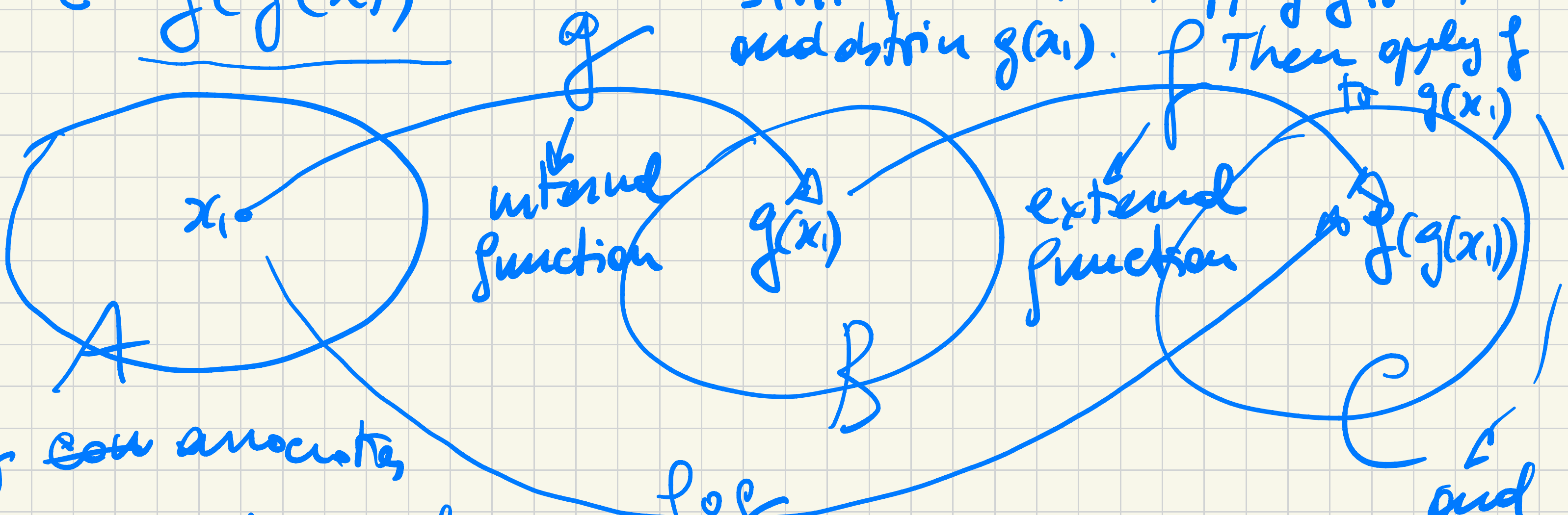
$R_f = [1,5]$

$f : [1,6] \to \mathbb{R}$

$f(1) = 1$, $f(2) = 3$, $f(3) = 4.9$, $f(5.5) = 2.9$

DEF Given two functions $g: A \to B$, $f: B \to C$, the composite function $f \circ g : A \to C$ associates to each $x \in A$ the element $z \in C$ such that $z = f(g(x))$

Start from $x_1 \in A$, apply g to x_1 and obtain $g(x_1)$. Then apply f to $g(x_1)$ and get $f(g(x_1))$



$f \circ g$ ~~can~~ associates to each $x \in A$ an element z in C, $f(g(x))$, without mentioning the set B

$A = B = C = \mathbb{R}$, $\quad g(x) = 2x - 7$, $\quad \boxed{f(x) = x^2 + 4x + 9}$

$$(f \circ g)(x) = f(g(x)) = f(2x-7) \overset{?}{=} (2x-7)^2 + 4(2x-7) + 9$$

$$= 4x^2 + 49 - 28x + 8x - 28 + 9$$

$$= \underline{4x^2 - 20x + 30}$$

$$f(5) = 25 + 20 - 9 = 36$$

$$f(1) = 1 + 4 - 9 = -4$$

$$f(1 + 3x) = (1+3x)^2 + 4(1+3x) + 9$$

$$(g \circ f)(x) = g(f(x)) = g(x^2 + 4x + 9) = 2(x^2 + 4x + 9) - 7$$

$$= 2x^2 + 8x + 11$$

$$\boxed{h(x) = (4x + 3 - x^2)^7} = f(g(x)) \text{ with } g(x) = 4x + 3 - x^2$$

$$f(x) = x^7$$

NATURAL DOMAIN

If for a function, the domain is not specified, ~~thus~~ but only the expression is given, then the function is considered defined on its "natural domain", that is the ~~s~~ set of real numbers for which $f(x)$ can be evaluated.

$$f(x) = \frac{\sqrt{x+3}}{x-5}$$

$$f(-6) = \frac{\sqrt{-6+3}}{-6-5} = \frac{\sqrt{-3}}{-11}$$

$$f(5) = \frac{\sqrt{5+3}}{5-5} = \frac{\sqrt{8}}{0}$$

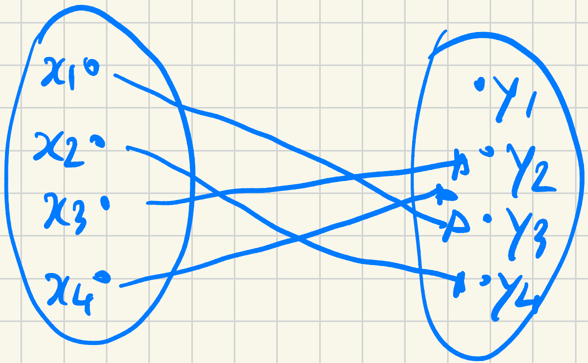
Natural domain of f

$$= \{x \in \mathbb{R} : x+3 \geq 0 \text{ and } x-5 \neq 0\}$$

$$= \{x \in \mathbb{R} : x \geq -3,\ x \neq 5\}$$

$$= [-3, 5) \cup (5, +\infty)$$

DEF Given a $f: A \to B$, the <u>range of f</u>, or <u>Image of f</u>, is the set $R_f = \{ y \in B :$ there exists at least one $x \in A$ s.t. $f(x) = y \} \subseteq B$

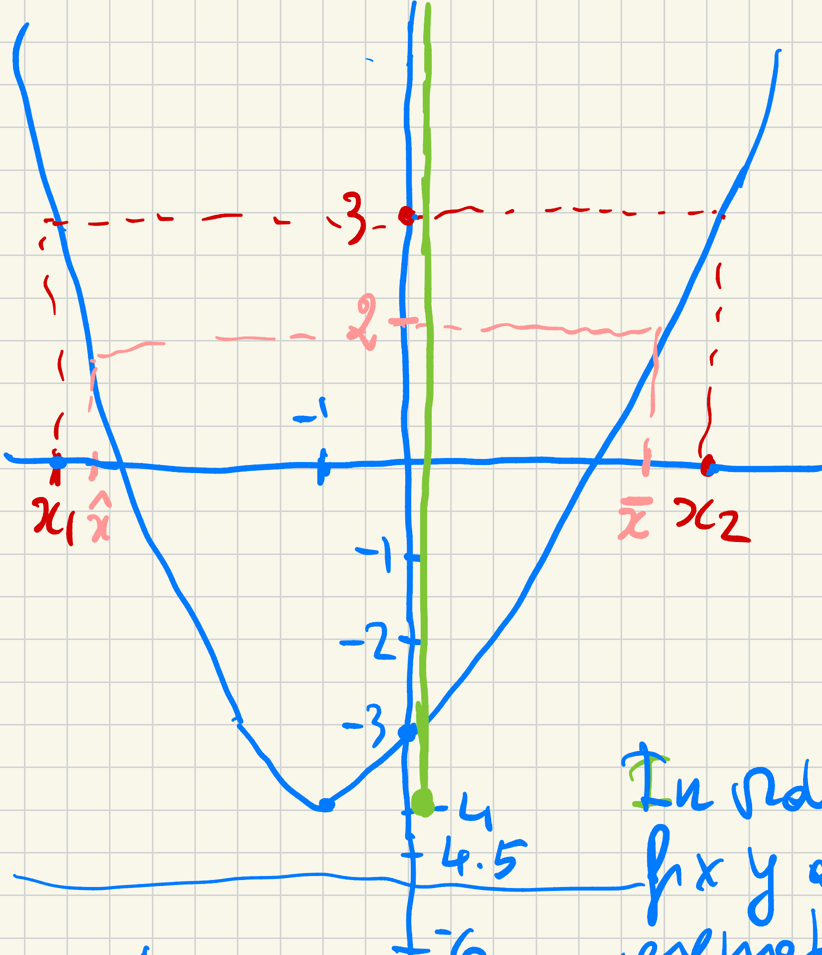


$y_3 \in R_f$? $f(x_1) = y_3$, hence

$y_3 \in R_f$. $f(x_2) = y_4$, hence

$y_4 \in R_f$: $R_f = \{ y_2, y_3, y_4 \}$

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 + 2x - 3$



$f:\mathbb{R}\to\mathbb{R},\quad f(x)=x^2+2x-3$

$R_f = ?\qquad 3\in R_f$

$f(\bar{x})=2,\ f(\hat{x})=2,\ 2\in R_f$

$f(0)=-3,\ -3\in R_f$

$-4\in R$

the same condition

$R_f=[-4,+\infty)$

In order to determine R_f with algebra, fix y and look for solutions to the equation $f(x)=y \iff x^2+2x-3=y$

$1\cdot x^2+2x-3-y=0 \iff \boxed{y\geq -4}$

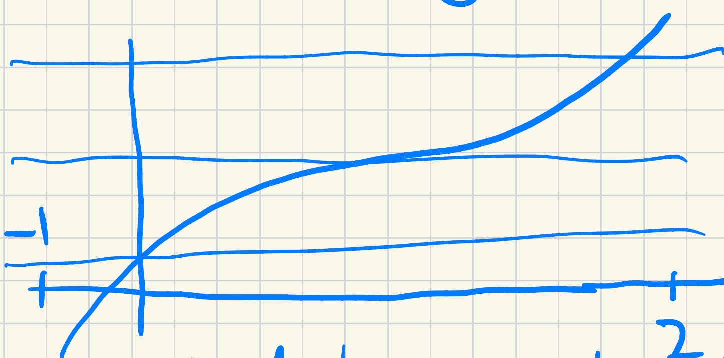
$\Delta = 4-4\cdot 1\cdot(-3-y)$
$= 16+4y\geq 0$

DEF Given $f: A \to B$, f is said to be <u>one-to-one</u>, or <u>injective</u>, or <u>onto</u>, if for each x_1, x_2 in A with $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$.

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 + 2x - 3$ is not 1-to-1

The f of ~~#2~~ 2 pages ago is not 1-to-1

" " 6 " " is not 1-to-1



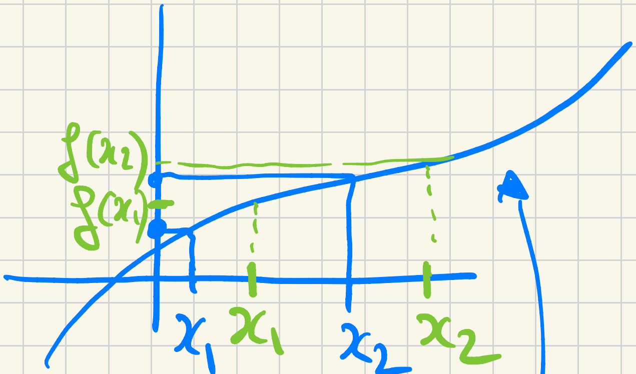
$f: [-1, 7] \to \mathbb{R}$ is 1-to-1

Horizontal line test: A real valued function of a real variable is 1-to-1 $\iff$ each horizontal line touches the graph in zero points or in one point.

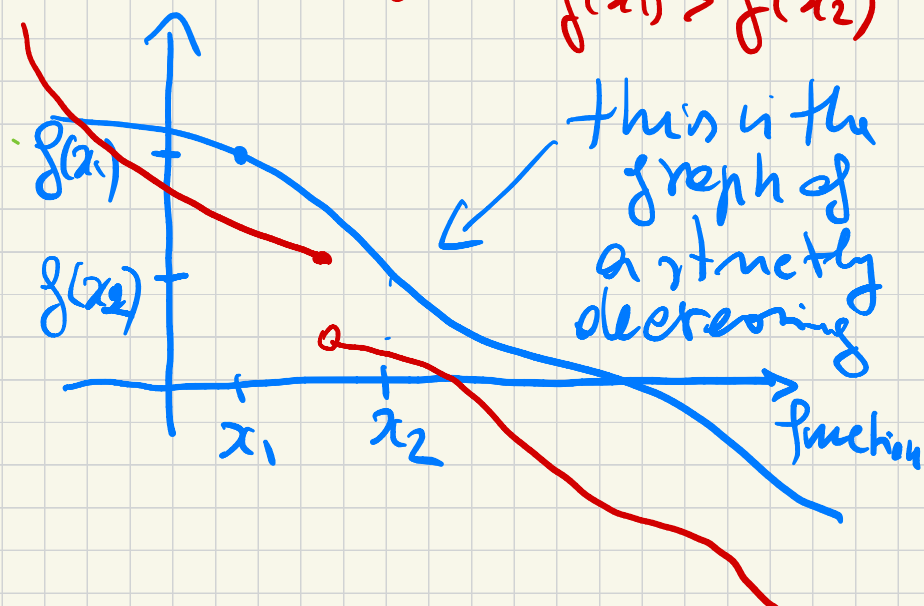
DEF A function $f: A \to \mathbb{R}$ ($A \subseteq \mathbb{R}$) is said to be monotone strictly increasing if for each x_1, x_2 in A with $x_1 < x_2$ we have $f(x_1) < f(x_2)$

decreasing

$f(x_1) > f(x_2)$



this is the graph of a monotone strictly increasing function



this is the graph of a strictly decreasing

Useful property: If $f: A \to \mathbb{R}$ $(A \subseteq \mathbb{R})$ is monotone strictly increasing or monotone strictly decreasing, then f is 1-to-1. That is strict monotonicity implies 1-to-1

PROOF Say f is monotone strictly increasing. I want to prove that f is 1-to-1, that is if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

Take $x_1 \neq x_2$. Then 2 cases are possible

$x_1 < x_2$

$f(x_1) < f(x_2)$, hence

$f(x_1) \neq f(x_2)$

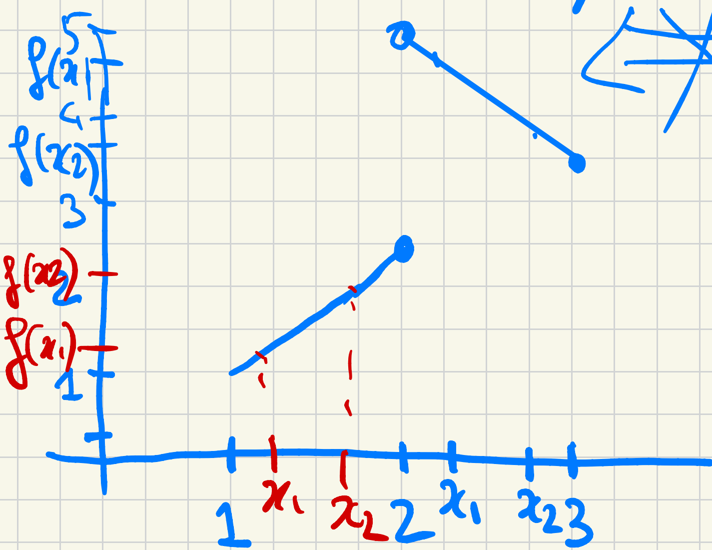
$x_1 > x_2$

$f(x_1) > f(x_2)$, hence

$f(x_1) \neq f(x_2)$ ∎

Strict monotonicity $\Rightarrow$ 1-to-1

$\not\Leftarrow$



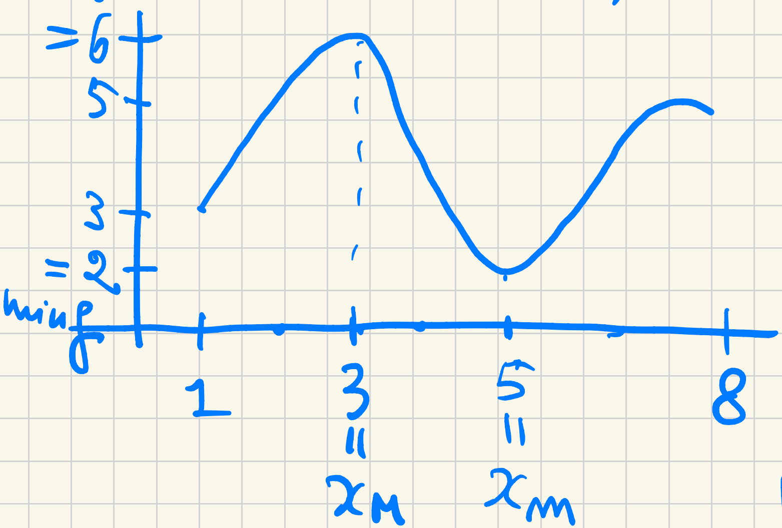
$f:[1,3]\to\mathbb{R}$ is 1-to-1 because of the horizontal line test.

f is not monotone strictly increasing

f is not monotone strictly decreasing

f is 1-to-1, but not strictly monotone increasing nor decreasing

DEF For a function $f: A \to \mathbb{R}$, $x_M \in A$ is said to be a <u>maximum point</u> if $f(x_M) \geq f(x)$ for each $x \in A$, $x \neq x_M$; $f(x_M) = \max f$



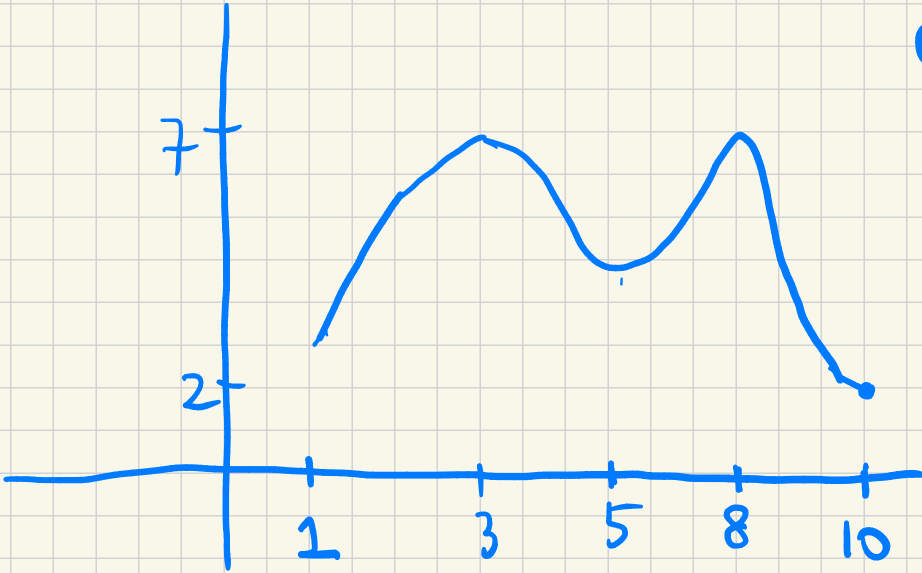
$f: [1, 8] \to \mathbb{R}$

$f(3) = 6$, hence $x_M = 3$

$6 = \max f$

$x_m \in A$ is said to be a <u>minimum point</u> if $f(x_m) \leq f(x)$ for each $x \in A$, $x \neq x_m$

$\min f = f(x_m)$

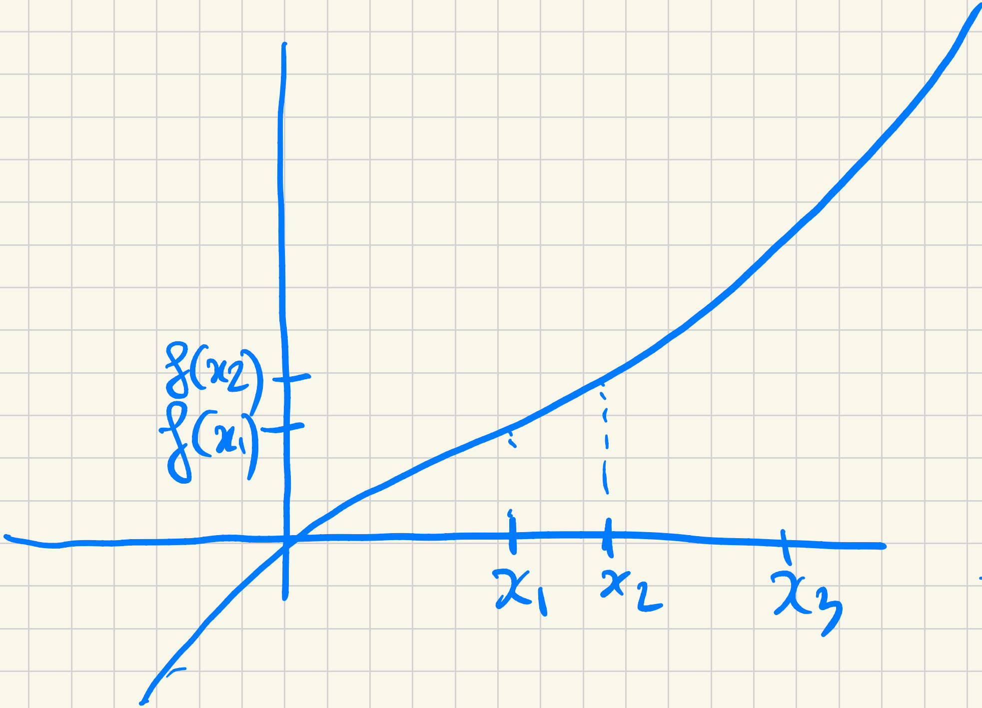


$f : [1, 10] \to \mathbb{R}$

$x_m = 10$, $\min f = 2$

$x = 3$, $x = 8$ are both maximum point for f

$\max f = 7$



$f: \mathbb{R} \to \mathbb{R}$

What is a max point for this function?

There exists no max point for this function, and no minimum point exists either.