LECTURE 4 (Thursday, Sept. 10, 2026)

DEF Given a function $f:[a,b]\to\mathbb{R}$, f continuous, a <u>primitive of f</u> is $F:[a,b]\to\mathbb{R}$ which is differentiable and such that $F'(x) = f(x)$ for each $x\in A$

Hence, a primitive of f is just a function whose derivative coincides with f.

Example For $f(x)=3$ (the function with constant value 3), a primitive is $F(x)=3x$, as $F'(x)=3=f(x)$. But that is not the unique primitive of f, as also $G(x)=3x+12$ is a primitive of f and, more generally, $H(x)=3x+K$ is a primitive of f for each $K\in\mathbb{R}$

The symbol $\int f(x)dx$ is said indefinite integral of f and denotes an arbitrary primitive of f. Thus the writing

$$\int f(x)\,dx = F(x)$$

indicates that F is a primitive of f, but sometimes the writing $\int f(x)dx = F(x)+K$ is employed to remind that $F(x)+K$ is a primitive of f for each $K \in \mathbb{R}$

Example $\int x\,dx = \frac{1}{2}x^2$, $\int e^x dx = e^x$

Table of primitives

$f(x)$	K	$\overbrace{x^a}^{\text{if } a \neq -1}$	$\frac{1}{x}$	a^x	e^x
$\int f(x)$	Kx	$\frac{1}{a+1}x^{a+1}$	$\ln\lvert x\rvert$	$\frac{a^x}{\ln a}$	e^x

Properties of indefinite integrals : $\int a f(x)\,dx = a\int f(x)\,dx$

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Example $\int \left(5x^2 - 4x + 4 + \frac{7}{x} - 6e^x + 8e^{2x} + \frac{9}{x^2} + 3e^{2x}\right) dx$

$$= \frac{5}{3}x^3 - 2x^2 + 4x + 7\ln|x| - 6e^x + 4e^{2x} - \frac{9}{x} + \frac{3}{2}e^{2x}$$

INTEGRATION BY PARTS

The rule to differentiate a product states

$$(f(x)\cdot g(x))' = f'(x)g(x) + f(x)g'(x)$$

Taking the integral of both sides yields

$$\int (f(x)g(x))'\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

$$f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

hence $\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx$ FORMULA OF INTEGRATION BY PARTS

This means that when trying to calculate $\int f'(x) g(x) dx$, the difficulty can be moved to $\int f(x) g'(x)$. If the second integral is simpler than the first, then the formula of integration by parts facilitates the task

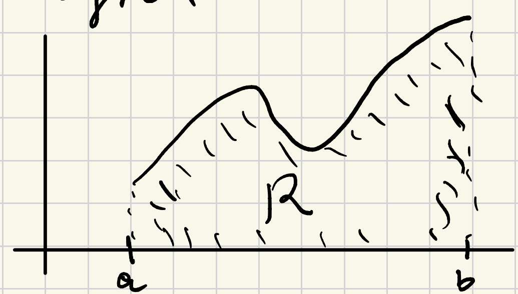
Example $\int \underbrace{x}_{g(x)} \underbrace{e^x}_{f'(x)} dx = e^x \cdot x - \int e^x \cdot 1 dx = e^x x - e^x$, hence $F(x) = e^x \cdot x - e^x$ is a primitive of $f(x) = e^x \cdot x$

DEFINITE INTEGRALS

Let $f : [a,b] \to \mathbb{R}$, be continuous in $[a,b]$ and such that $f(x) \geq 0$ for each $x \in [a,b]$. Consider the region

$R = \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq f(x)\}$

its area is denoted with $\int_a^b f(x) dx$, which is read definite integral of f in the interval $[a,b]$



Although $\int f(x)\,dx$ and $\int_a^b f(x)\,dx$ are similar symbols, they are quite different objects, as the first is a function (a primitive of f, or the set of all the functions which are primitives of f), and the second is a number (the area of region R)

The following theorem indicates a method to determine $\int_a^b f(x)\,dx$

THEOREM (fundamental theorem of calculus[1])

Hp: $f : [a,b] \to \mathbb{R}$, f is continuous in $[a,b]$, F is an arbitrary primitive of f

Ts: $\int_a^b f(x)\,dx = F(b) - F(a) = [F(x)]_a^b$

Thus a definite integral can be computed as the variation of any primitive of f in $[a,b]$.

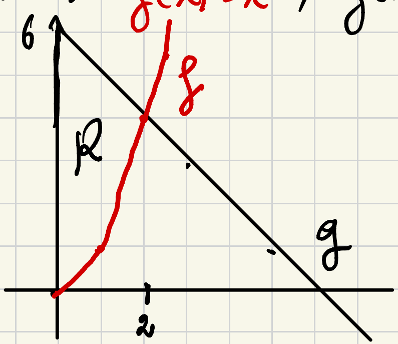
Examples $\int_1^4 x\,dx = \left[\frac{1}{2}x^2\right]_1^4 = 8 - \frac{1}{2} = 7.5$

$\int_1^e \ln x\,dx$ can be evaluated after determining a primitive of $\ln x$. This can be done by parts as follows

$$\int \ln x\,dx = \int \underset{f'}{1} \cdot \underset{g}{\ln x}\,dx = x\ln x - \int x \cdot \frac{1}{x}\,dx = x\ln x - \int 1 \cdot dx$$
$$= x\ln x - x$$

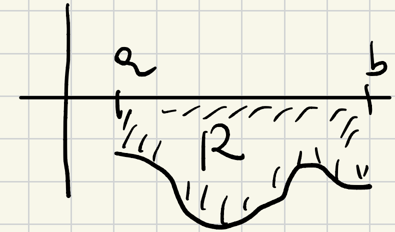
Hence $x\ln x - x$ is a primitive of $\ln x$ and $\int_1^e \ln x\,dx = [x\ln x - x]_1^e = 1$

Consider the functions $f(x) = x^2$, $g(x) = 6 - x$, with the following graphs for $x \geq 0$



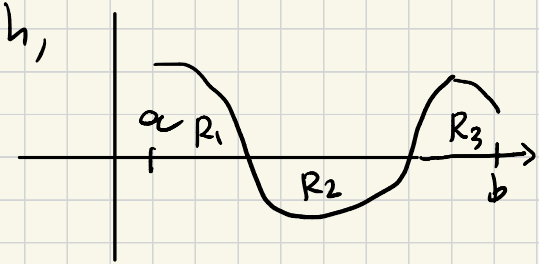
Suppose we want to evaluate the area of the region $R = \{(x,y) \in \mathbb{R}^2 :$
$0 \leq x \leq 2,\ f(x) \leq y \leq g(x)\}$

If $f(x)<0$ for each $x\in[a,b]$, then $\int_a^b f(x)\,dx<0$ and it is equal to the negative of the area of the region R



For f with the following graph,

$$\int_a^b f(x)\,dx = \text{area of } R_1 - \text{area of } R_2 + \text{area of } R_3$$

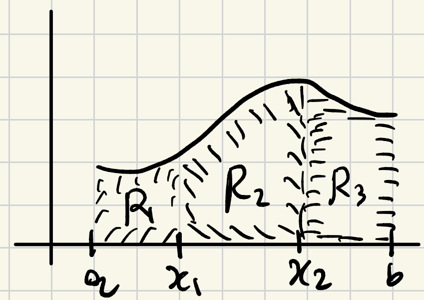


Given $f:[a,b]\to\mathbb{R}$, f continuous in $[a,b]$, the integral function of f is $G:[a,b]\to\mathbb{R}$, $G(x)=\int_a^x f(t)\,dt$, in which x is the right extreme of the interval of integration

$G(x_1)=$ area of R_1

$G(x_2)=$ area of $R_1\cup R_2$

$G(b)=$ area of $R_1\cup R_2\cup R_3$



THEOREM (fundamental theorem of calculus 2)

Hp: $f:[a,b]\to\mathbb{R}$, f is continuous in $[a,b]$

Ts: The integral function $G(x)=\int_a^x f(t)dt$ is differentiable in $[a,b]$, with $G'(x)=f(x)$ for each $x\in[a,b]$.

This theorem allows to prove the equality $\int_a^b f(x)\,dx = F(b)-F(a)$, in which F is an arbitrary primitive of f. Precisely, since G and F are both primitives of f, it follows that there is a k such that $G(x)=F(x)+k$ for each $x\in[a,b]$. Moreover, using the definition of the integral function G of f, it follows that

$$G(b)=\int_a^b f(t)dt = G(b)-G(a) \quad \text{since } G(a)=\int_a^a f(t)dt=0$$

$$G(b)-G(a)=F(b)+k-F(a)-k=F(b)-F(a), \text{ hence } \int_a^b f(t)dt=F(b)-F(a)$$

Properties of definite integrals

- $\int_a^a f(x)\,dx = 0$, $\quad \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ for $a < b$

- $\int_a^b k f(x)\,dx = k \int_a^b f(x)\,dx$, $\quad \int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$