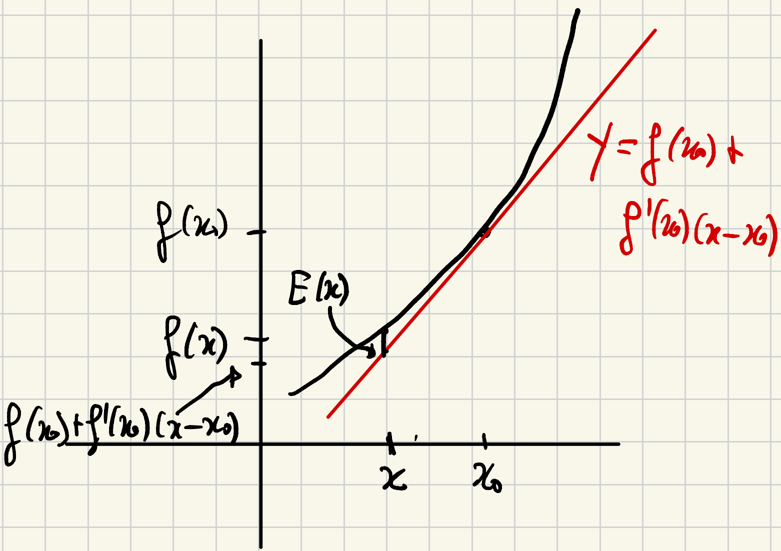
LECTURE (Wednesday, Sept. 9, 2026)

If f is differentiable, then the tangent line at the graph of f at $(x_0, f(x_0))$ has equation $y = f(x_0) + f'(x_0)(x - x_0)$ and the expression $f(x_0) + f'(x_0)(x-x_0)$ is a good approximation for $f(x)$ when x is close to x_0.

In details, the red line is the tangent line at $(x_0, f(x_0))$ and for $x \neq x_0$ it can be used to approx $f(x)$. In doing so, the approximation error is

$$E(x) = f(x) - [f(x_0) + f'(x_0)(x - x_0)],$$

and the closer is $E(x)$ to 0, the better is the approximation, that is the closer is $f(x_0) + f'(x_0)(x-x_0)$ to $f(x)$



It is possible to prove that $\lim_{x \to x_0} \frac{E(x)}{x-x_0} = 0$, which means that when x is close to x_0, $E(x)$ is much closer to 0 than $x-x_0$. It is immediate that $x-x_0$ is close to 0 when x is close to x_0, hence $E(x)$ is even closer to 0, which indicates that the approx of $f(x)$ through $f(x_0)+f'(x_0)(x-x_0)$ is a good approximation.

This is written as $f(x) \simeq f(x_0) + f'(x_0)(x-x_0)$ for x close to x_0

$$\text{or } \underbrace{f(x)-f(x_0)}_{\Delta f} \simeq f'(x_0)\underbrace{(x-x_0)}_{\Delta x} \quad \text{for } x \text{ close to } x_0$$

$$\Delta f \simeq \underbrace{f'(x_0)\Delta x}_{\substack{\text{differential} \\ \text{of } f}} \quad \text{for } \Delta x \text{ close to } 0$$

This approximate equality says that the variation of f as a consequence of a change in the indep. variable from x_0 to x is about equal to $f'(x_0)$ times the change in x.

Example $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 6x + 5$, $x_0 = 4$, $f'(4) = 2$, $x = 4.09$

Then $\Delta f \simeq 2 \cdot \Delta x = 2 \cdot 0.09 = 0.18$.

Thus if the independent variable changes from 4 to 4.09, the value of f changes by about 0.18 (the true value, $f(4.09) - f(4)$, is 0.1881)

MONOTONICITY AND DERIVATIVES

DEF $f: A \to \mathbb{R}$ ($A \subseteq \mathbb{R}$) is said to be monotone, strictly increasing (decreasing) if for each x_1, x_2 in A with $x_1 < x_2$ we have $f(x_1) < f(x_2)$ ($f(x_1) > f(x_2)$)

The following theorem indicates a method to determine the monotonicity of f based on f's derivative

THEOREM (monotonicity test) Hp: $f: A \to \mathbb{R}$ is differentiable in the interval $(a,b) \subseteq A$

Ts: (i) If $f'(x) > 0$ for each $x \in (a,b)$, then f is monotone strictly increasing in (a,b)

(ii) If $f'(x) < 0$ for each $x \in (a,b)$, then f is monotone strictly decreasing in (a,b)

This theorem reveals that it is possible to learn the monotonicity of a differentiable function if it is possible to determine the sign of the derivative

Example $f:[-2,3]\to\mathbb{R}$, $f(x)=\frac{3}{2}x^4-2x^3-6x^2+4$ has derivative

$f'(x)=6x^3-6x^2-12x=6x(x^2-x-2)$ and

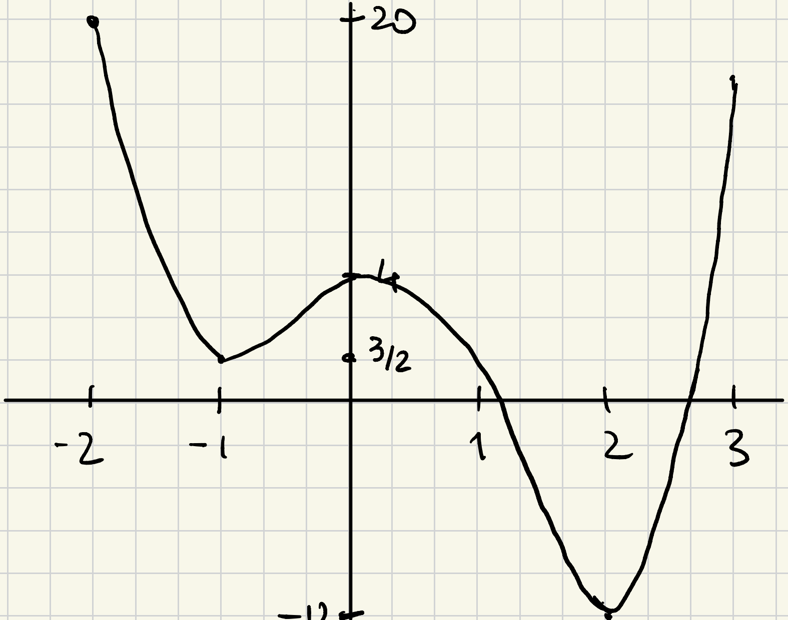
	-2 to -1	-1 to 0	0 to 1	1 to 2	2 to 3
$6x$	$-$	$-$	$+$	$+$	$+$
x^2-x-2	$+$	$-$	$-$	$-$	$+$
$f'(x)$	$-$	$+$	$-$	$-$	$+$

$f'(x)<0$ for each $x\in(-2,-1)$, f is strictly decreasing in $[-2,-1]$

$f'(x)>0$ for each $x\in(-1,0)$, thus f is strictly increasing in $[-1,0]$

$f'(x)<0$ for each $x\in(0,2)$, thus f is strictly decreasing in $[0,2]$

$f'(x)>0$ for each $x\in(2,3)$, thus f is strictly increasing in $[2,3]$



$f(-2) = 20$, $f(-1) = 3/2$, $f(2) = -12$
$f(3) = 17.5$

Hence $x_M = -2$, $\max f = 20$

$x_m = 2$, $\min f = -12$

DEF Given $f: A \to \mathbb{R}$, x_M (x_m) is said to be a <u>global max (min) point for f</u> if

$f(x_M) \geq f(x)$ for each $x \in A$;
$f(x_m) \leq f(x)$

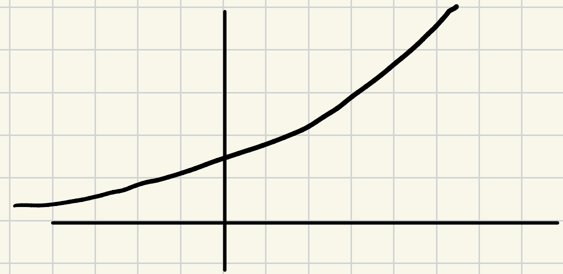
$\max f = f(x_M)$ is the maximum (minimum) value of f

$\min f = f(x_m)$

The points where the monotonicity of f changes are points such that $f'(x) = 0$; hence the tangent line at those points is horizontal.

If x_0 is such that $f'(x_0) = 0$, then x_0 is said to be a <u>critical point of f</u>, or a <u>stationary point</u>

In the example of the last two pages, x_M and x_m exist, but that is not necessarily always the case. For instance, $f(x) = 2^x$, $f: \mathbb{R} \to \mathbb{R}$ has no global max point nor global min point.



However, there exists a theorem which indicates sufficient conditions for the existence of x_M, x_m

THEOREM (Extreme Value Theorem)

Hp: $f: [a,b] \to \mathbb{R}$ is continuous in $[a,b]$

Ts: There exists x_M and there exists x_m

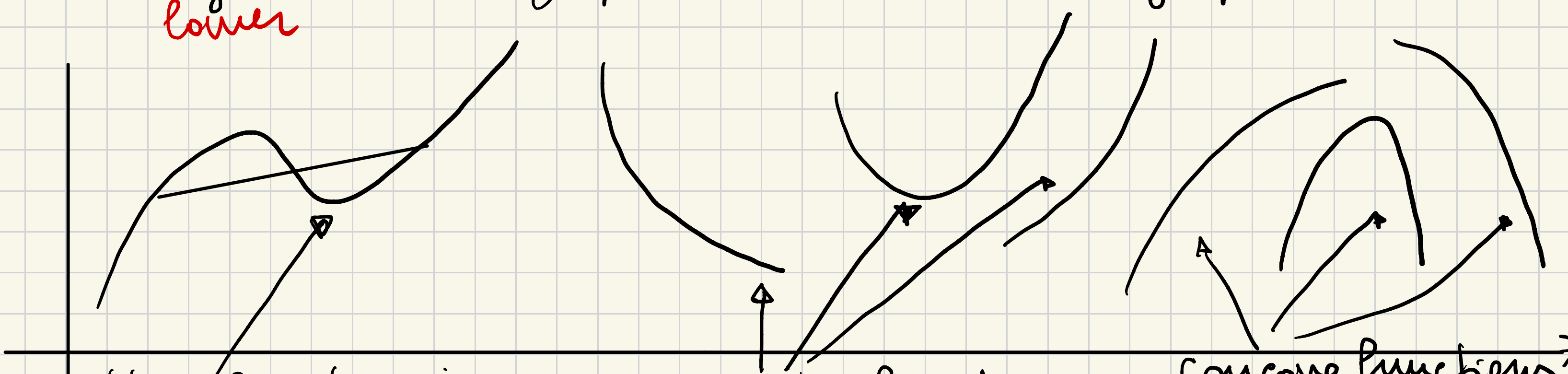
The exponential function in this page violates the assumptions of this theorem as its domain $\mathbb{R} = (-\infty, +\infty)$ is not a bounded interval.

The function of the two previous pages satisfies the assumptions of the

EVT as its domain is $[a,b] = [-2,3]$ and f is continuous in $[-2,3]$; indeed x_m and x_M exist.

CONCAVE, CONVEX FUNCTIONS

DEF Given $f: A \to \mathbb{R}$ (A is an interval), f is said to be concave
be <u>convex</u> if for each two points on the graph of f the segment connecting the two points is entirely higher than the graph or coincides with the graph
lower



DEF If $f:(a,b)\to\mathbb{R}$ is differentiable in (a,b) and f' is differentiable in (a,b), then f is said to be twice differentiable in (a,b) and the derivative of f' is called second derivative of f, denoted f''.

Example $f:\mathbb{R}\to\mathbb{R}$, $f(x)=7x^9+3e^x$, $f'(x)=63x^8+3e^x$, $f''(x)=504x^7+3e^x$

THEOREM (concavity/convexity test)

Hp: $f:A\to\mathbb{R}$ is twice differentiable in $(a,b)\subseteq A$

Ts: (i) f is concave in (a,b) $\iff$ f' is monotone decreasing in (a,b) $\iff$ $f''(x)\le 0$ for each $x\in(a,b)$

(ii) f is convex in (a,b) $\iff$ f' is monotone increasing in (a,b) $\iff$ $f''(x)\ge 0$ for each $x\in(a,b)$.

Example $f:(0,+\infty)\to\mathbb{R}$, $f(x)=7\ln x-2x$ is concave as
$f'(x)=\frac{7}{x}-2$, $f''(x)=-\frac{7}{x^2}<0$ for each $x\in(0,+\infty)$

$g(x)=\frac{1}{2}x-\sqrt{x+1}$, $g:[-1,+\infty)\to\mathbb{R}$ is convex as $g'(x)=\frac{1}{2}-\frac{1}{2\sqrt{x+1}}$,
$g''(x)=\frac{1}{4(x+1)^{3/2}}>0$ for each $x>-1$

AN IMPORTANT PROPERTY OF CONCAVE/CONVEX FUNCTIONS

Given $f:A\to\mathbb{R}$ convex (concave), the following inequality holds

$$f(x)\geq f(x_0)+f'(x_0)(x-x_0) \quad \text{for each } x\in A,\ x_0\in A$$
$$f(x)\leq f(x_0)+f'(x_0)(x-x_0) \quad \text{for each } x\in A,\ x_0\in A$$

Then suppose x_0 is a stationary point for f, that is $f'(x_0)=0$. The inequality above boils down to $f(x)\geq f(x_0)$ for each $x\in A$, hence $x_m=x_0$

$f(x)\leq f(x_0)$ for each $x\in A$, hence $x_M=x_0$

Therefore in order to maximize a concave function it suffices to find a stationary point, and in order to minimize a convex function it suffices to identify a stationary point