LECTURE 2 (Tuesday, Sept. 8, 2026)

THEOREM (IVT) Hp: $f:[a,b]\to\mathbb{R}$ is continuous in $[a,b]$ and $f(a)\neq f(b)$

Ts: For each y_0 between $f(a)$ and $f(b)$ there exists at least one $x_0\in(a,b)$ such that $f(x_0)=y_0$; if f is strictly monotone, then x_0 is unique

From the IVT the following corollary is obtained

COROLLARY TO THE IVT Hp: $f:[a,b]\to\mathbb{R}$ is continuous in $[a,b]$ and $\left.\begin{matrix} f(a)<0,\ f(b)>0 \\ \text{OR} \\ f(a)>0,\ f(b)<0 \end{matrix}\right\} f(a)f(b)<0$

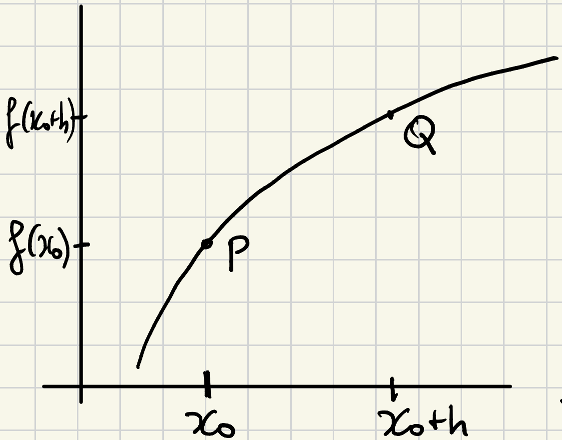
Ts: There exists at least one $x_0\in(a,b)$ such that $f(x_0)=0$ (that is, x_0 is a solution to the equation $f(x)=0$); if f is strictly monotone, then x_0 is unique

Example $2^x + 7\sqrt{x+1} = 18$ is an equation for which there exists no formula for a solution. An equivalent equation is $2^x + 7\sqrt{x+1} - 18 = 0$, and set $f(x) = 2^x + 7\sqrt{x+1} - 18$; this is a continuous function over its natural domain, the interval $[-1, +\infty)$.

In order to apply the Corollary to the IVT to prove that a solution exists, it is necessary to determine $[a,b]$ such that $f(a) \cdot f(b) < 0$. After some trial and error we find $f(2) < 0$, $f(3) > 0$, hence the corollary implies that a solution to the equation $f(x) = 0$ exists in $(2,3)$. Moreover, f is strictly increasing as 2^x is strictly increasing, $\sqrt{x+1}$ is strictly increasing, -18 is constant. Hence a unique solution exists.

DIFFERENTIAL CALCULUS

Consider $f:(a,b)\to\mathbb{R}$, $x_0\in(a,b)$ and the line through $P=(x_0,f(x_0))$
and $Q=(x_0+h, f(x_0+h))$



This line has slope

$$\underbrace{\frac{f(x_0+h)-f(x_0)}{h}}_{\text{incremental quotient, a function of } h}$$

which can be viewed as $\frac{\Delta f}{\Delta x}$

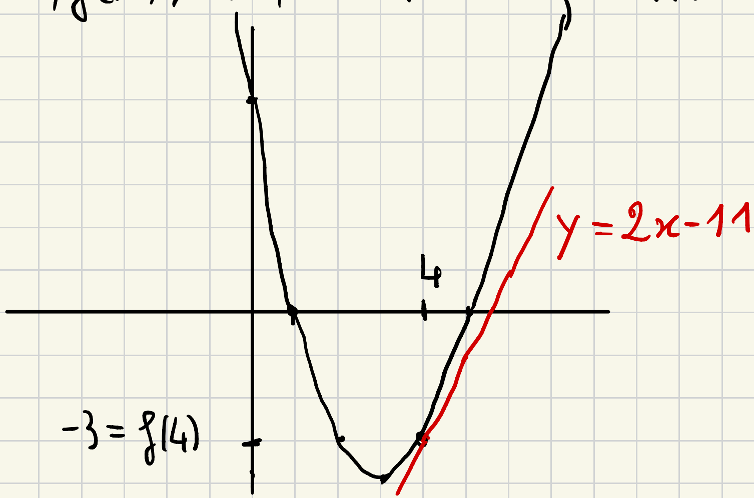
If $\lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$ exists and is finite, then its value is called <u>derivative of f at x_0</u>, denoted $f'(x_0)$, and f is said to be <u>differentiable at x_0</u>

Example $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x^2-6x+5$, $x_0=4$, I want to determine $f'(4)$. The incremental quotient at $x_0=4$ is $\frac{f(4+h)-f(4)}{h}=$

$$= \frac{(4+h)^2 - 6(4+h) + 5 - (-3)}{h} = \frac{16 + h^2 + 8h - 24 - 6h + 8}{h} = \frac{h^2 + 2h}{h} = h + 2$$

and $f'(4) = \lim_{h \to 0} (h+2) = 2$, thus $f'(4) = 2$

When f is differentiable at x_0, the line with equation $y = f(x_0) + f'(x_0)(x - x_0)$ is said tangent line to the graph of f at $(x_0, f(x_0))$. For the example above, the tangent line at $(x_0, f(x_0)) = (4, -3)$ has equation $y = -3 + 2(x-4) = 2x - 11$



If $f : (a,b) \to \mathbb{R}$ is differentiable at each $x_0 \in (a,b)$, then f is said to be differentiable in (a,b).

Notation: Sometimes the derivative is denoted with $\frac{df}{dx}$, or Df, or y', or $\frac{dy}{dx}$

DERIVATIVES OF ELEMENTARY FUNCTIONS

We first need to introduce a limit: $\lim_{x\to 0} (1+x)^{1/x}$. With a bit of work it can be proved that this limit exists and is about equal to 2.71; it is denoted with the symbol e.

Since $e>0$, $e\neq 1$, e can be used as a base for an exponential function and for a log function, that is there exists the exp. function with base e: $f(x)=e^x$, and the log function with base e, $g(x)=\log_e x = \ln x$ called "natural log functions".

The function $f(x)$ is more important than a generic exp function
" " $g(x)$ " " " " " log function

a constant function

A table of derivatives

$f(x)$	k	x	x^a	a^x	e^x	$\log_a x$	$\ln x$
$f'(x)$	0	1	$\underbrace{ax^{a-1}}_{\text{except if } a\in(0,1) \text{ and } x=0}$	$a^x \ln a$	e^x	$\dfrac{1}{x\cdot\ln a}$	$\dfrac{1}{x}$

These are the derivatives of the elementary functions, obtained by relying on the definition given three pages ago

THEOREM If f and g are differentiable, then

(i) $h(x) = af(x)$ is differentiable and $h'(x) = af'(x)$

(ii) $h(x) = f(x) \pm g(x)$ " " $h'(x) = f'(x) \pm g'(x)$

(iii) $h(x) = f(x)\cdot g(x)$ " " $h'(x) = f'(x)g(x) + f(x)g'(x)$

(iv) $h(x) = f(x)/g(x)$ " " $h'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

Examples

$h_1(x) = 3x^2+5$ has derivative $h_1'(x) = 3\cdot 2x + 0 = 6x$

$h_2(x) = 6 - 2\ln x$ " $h_2'(x) = -2\cdot\frac{1}{x} = -\frac{2}{x}$

$h_3(x) = 7e^x$ " $h_3'(x) = 7e^x$

$h_4(x) = -2x^5 + \frac{4}{9}x - 3\sqrt{x} + \frac{7}{x} + \frac{1}{15}\log_2 x + \frac{9}{16}3^x$ has derivative

$h_4'(x) = -10x^4 + \frac{4}{9} - \frac{3}{2\sqrt{x}} - \frac{7}{x^2} + \frac{1}{15}\cdot\frac{1}{x\ln 2} + \frac{9}{16}3^x\cdot\ln 3$

$h_5(x) = 5^x(x^4-3)$ has derivative $h_5'(x) = 5^x\cdot\ln 5\,(x^4-3) + 5^x\cdot 4x^3$

$h_6(x) = \frac{e^x-2}{3+4x^5}$ has derivative $h_6'(x) = \frac{e^x(3+4x^5) - (e^x-2)\,20x^4}{(3+4x^5)^2}$

If f and g are differentiable, then $h(x) = f(g(x))$ is differentiable and $h'(x) = g'(x)\, f'(g(x))$

Examples $h_7(x) = e^{3x+1} = f(g(x))$ with $g(x) = 3x+1$, $f(x) = e^x$

Then $h_7'(x) = \underbrace{3}_{g'(x)} \cdot \underbrace{e^{3x+1}}_{f'(g(x))}$

$h_8(x) = \ln(7x+11x^2) = f(g(x))$ with $g(x) = 7x+11x^2$, $f(x) = \ln x$

Then $h_8'(x) = \underbrace{(7+22x)}_{g'(x)} \cdot \underbrace{\frac{1}{7x+11x^2}}_{f'(g(x))}$