LECTURE 1 (Monday, Sept. 7, 2026)

A brief and unrigorous introduction to the notion of limit

Consider a function $f : \overset{\text{domain of } f}{A} \to \underset{\text{codomain of } f}{\mathbb{R}}$ $(A \subseteq \mathbb{R})$, $x_0 \in \mathbb{R}$

The writing $\lim_{x \to x_0} f(x) = L$ is read "limit of $f(x)$ as x tends to x_0 is equal to L"; $\lim_{x \to x_0} f(x)$ is about $f(x)$ when x is close to x_0 (hence $f(x)$ needs to be defined for x close to x_0), but is unaffected by $f(x_0)$, that is f when x is equal to x_0 does not matter

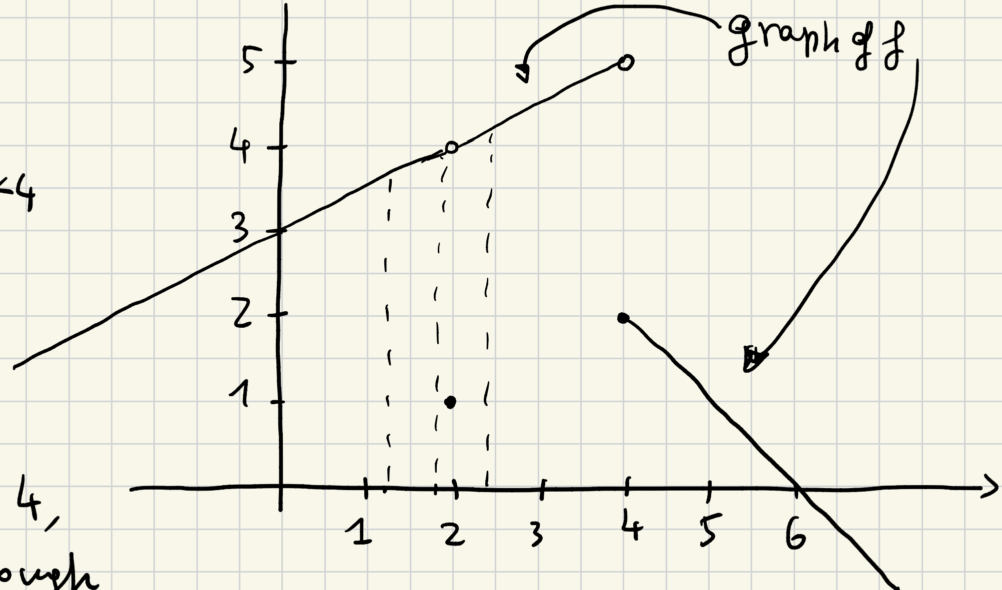
When $\lim_{x \to x_0} f(x) = L$, with $L \in \mathbb{R}$, the interpretation is that when x is close to x_0, $f(x)$ is close to L.

Example

$$f(x) = \begin{cases} \frac{1}{2}x+3 & \text{for } x<2 \\ 1 & \text{for } x=2 \\ \frac{1}{2}x+3 & \text{for } 2<x<4 \\ 6-x & \text{for } x\geq 4 \end{cases}$$

$f: \mathbb{R} \to \mathbb{R}$

A



As x tends to 2, $f(x)$ tends to 4, thus $\lim_{x\to 2} f(x) = 4$, even though $f(2)=1$

As x tends to 4, $f(x)$ tends to 2 or to 5 depending on whether x to 4 from values lower than 4 or from values greater than 4: $f(x)$ tends to 5 in the first case, tends to 2 in the second case and then we write $\lim_{x\to 4^-} f(x) = \lim_{x\uparrow 4} f(x) = 5$ and $\lim_{x\to 4^+} f(x) = \lim_{x\downarrow 4} f(x) = 2$. These are called one-sided limits.

$\lim_{x\to x_0} f(x)$ (bilateral limits) exists if and only if the two one-sided limits are equal. For the function in the example $\lim_{x\to 4} f(x)$ does not exist

DEF A function $f: A \to \mathbb{R}$ ($A \subseteq \mathbb{R}$) is said to be <u>continuous at $x_0 \in A$</u> if and only if $\lim_{x \to x_0} f(x) = f(x_0)$.

The definition is satisfied if and only if $\lim_{x \to x_0} f(x)$ happens to be equal to $f(x_0)$. If that is the case, then when x is close to x_0 we have that $f(x)$ is close to $f(x_0)$

The function of the previous page is not continuous at $x_0 = 2$ as $\lim_{x \to 2} f(x) = 4 \neq f(2) = 1$. Likewise, such a function is not continuous at $x = 4$ because $\lim_{x \to 4} f(x)$ does not exist. At each $x_0 \neq 2$, $x_0 \neq 4$, the function of the previous page is continuous

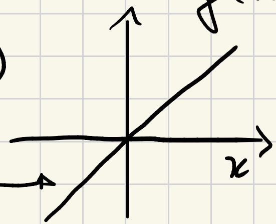
When f is continuous at x_0, it is simple to evaluate $\lim_{x \to x_0} f(x)$ as it is equal to $f(x_0)$

DEF A function $f : A \to \mathbb{R}$ ($A \subseteq \mathbb{R}$) is said to be <u>continuous in A</u> if and only if it is continuous at each $x_0 \in A$

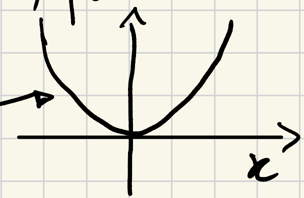
Several well-known functions (at least from the high school) are continuous, as illustrated in the following

POWER FUNCTIONS $f(x) = x^a$ with $a \in \mathbb{R}$, for instance

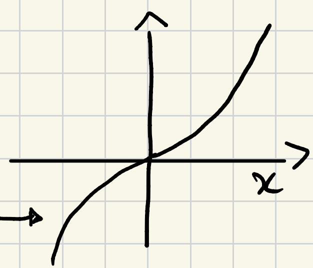
$f(x) = x^1$ ($a=1$)



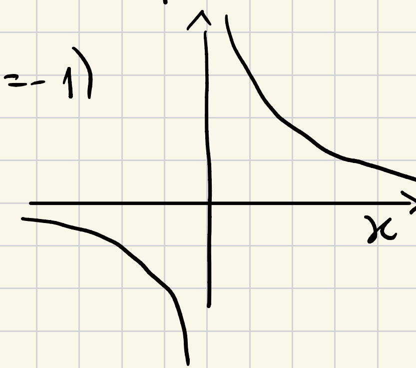
$f(x) = x^2$ ($a=2$)



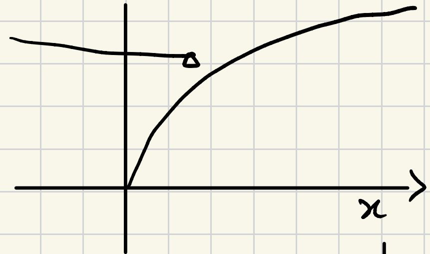
$f(x) = x^3$ ($a=3$)



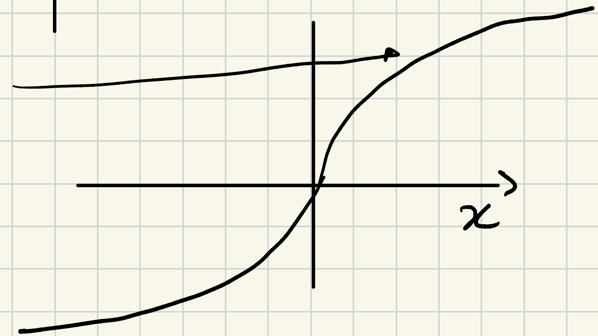
$f(x) = x^{-1}$ ($a=-1$)



$f(x) = \sqrt{x} = x^{1/2}$ $(a = 1/2)$



$f(x) = \sqrt[3]{x} = x^{1/3}$ $(a = 1/3)$



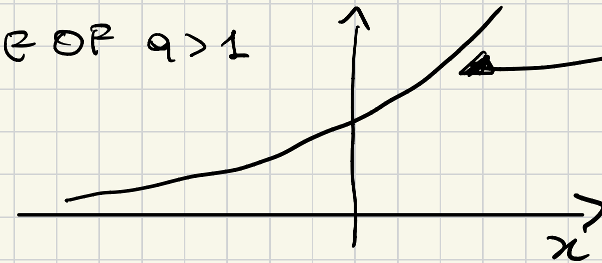
Given a function f, its natural domain is the set of all x such that $f(x)$ can be evaluated. For $f(x) = x^a$, the natural domain is $\mathbb{R}$ if the exponent a is integer and positive, it is $(-\infty, 0) \cup (0, +\infty)$ if the exponent a is a negative integer, is $[0, +\infty)$ if $a = \frac{1}{2}$, is $\mathbb{R}$ if $a = 1/3$

Each power function f is continuous in its natural domain

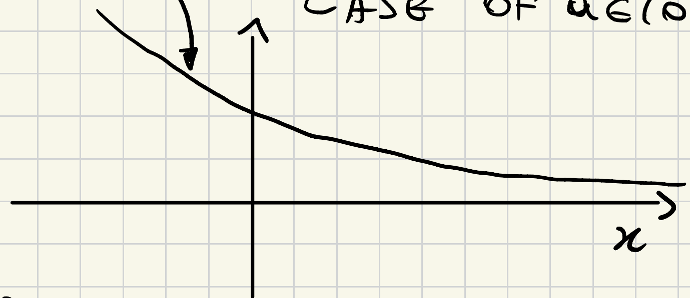
Hence, for instance, $\lim_{x\to 3} x^5 = 3^5$, $\lim_{x\to 3} \frac{1}{x} = \frac{1}{3}$, $\lim_{x\to 3} \sqrt{x} = \sqrt{3}$

EXPONENTIAL FUNCTIONS $f(x) = a^x$ with $a>0$, $a \neq 1$

CASE OF $a>1$



CASE OF $a \in (0,1)$



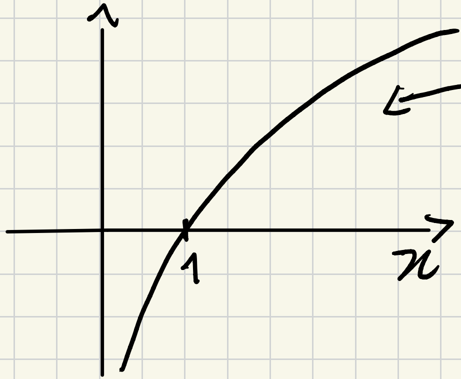
For each $a>0$, $a\neq 1$, the natural domain of $f(x)=a^x$ is $\mathbb{R}$, and f is continuous in $\mathbb{R}$. Hence

$$\lim_{x\to 4} 6^x = 6^4 = 6\cdot 6\cdot 6\cdot 6 = 1296$$

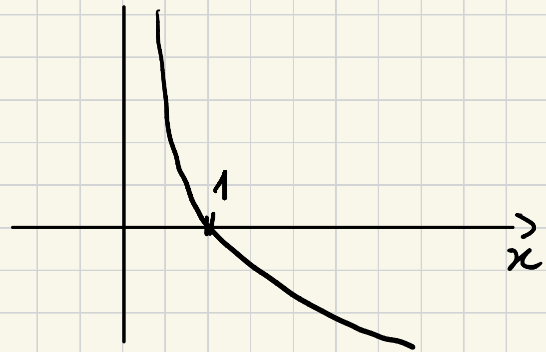
$$\lim_{x\to 2} \left(\frac{4}{11}\right)^x = \left(\frac{4}{11}\right)^2 = \frac{4}{11}\cdot\frac{4}{11} = \frac{16}{121}$$

LOG FUNCTIONS $f(x)=\log_a x$ with $a>0,\ a\neq 1$

CASE OF $a>1$



CASE OF $a\in(0,1)$



$\log_a x = y$ such that $a^y = x$

For each $a>0,\ a\neq 1$, the natural domain of $f(x)=\log_a x$ is $(0,+\infty)$, and f is continuous in $(0,+\infty)$.

Hence $\lim_{x\to 8}\log_2 x=\log_2 8=3$, $\lim_{x\to 6}\log_{3/4} x=\log_{3/4} 6$

If both f and g are continuous at x_0, then

- $f+g$ is continuous at x_0
- $f-g$ " " " "
- $f \cdot g$ " " " "
- f/g " " " " if $g(x_0) \neq 0$

the sum, difference, product, quotient of continuous functions is a continuous function

The function $f(x) = \dfrac{2^x(3-4x)}{5+x^6}$ is the difference, product, quotient of functions which are continuous in $\mathbb{R}$, hence f is continuous in $\mathbb{R}$

$f(x) = \dfrac{\sqrt{x}}{3-x}$ is continuous in its natural domain, which is $[0,3) \cup (3,+\infty)$

Given f and g, the composite function of f with g is such that

$h(x) = f(g(x))$

external function

internal function

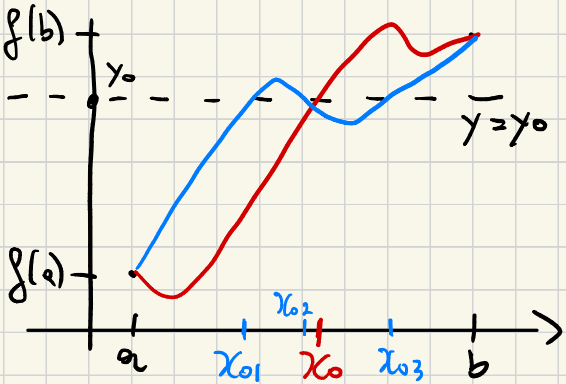
If $h(x) = \log_3(4+5x^2)$, then $h(x) = f(g(x))$ with $g(x) = 4+5x^2$, $f(x) = \log_3 x$

If g is continuous at x_0 and f is continuous at $g(x_0)$, then $h(x) = f(g(x))$ is continuous at x_0. In particular, $h(x) = \log_3(4+5x^2)$ is continuous in $\mathbb{R}$

An important property of continuous functions is described by the following theorem

THEOREM (Intermediate Value Theorem) Hp: $f : [a,b] \to \mathbb{R}$ is continuous in $[a,b]$ and $f(a) \neq f(b)$

Ts: For each y_0 between $f(a)$ and $f(b)$ there exists at least one $x_0 \in (a,b)$ such that $f(x_0) = y_0$



Given $(a, f(a))$, $(b, f(b))$, pick an arbitrary y_0 such that $f(a) < y_0 < f(b)$ and draw the line with equation $y = y_0$.

IVT says that as long as f is continuous in $[a,b]$, its graph intersects at least once the line $y = y_0$, and so x_0 is determined: the red graph and the blue graph are two examples of continuous graphs

If f satisfies the hypothesis of the IVT <u>and</u> moreover is monotone strictly increasing or monotone strictly decreasing, then there exists a unique $x_0 \in (a,b)$ such that $f(x_0) = y_0$. For the blue curve in the previous page there exist three x_0 such that $f(x_0) = y_0$, but that may happen only because the blue curve is not strictly monotone