DIFFERENTIATION RULES

If f and g are differentiable, then

$h(x) = af(x) + k$ is differentiable and $h'(x) = af'(x)$

$h(x) = f(x) \pm g(x)$ " " $h'(x) = f'(x) \pm g'(x)$

$h(x) = f(x) \cdot g(x)$ " " $h'(x) = f'(x)g(x) + f(x)g'(x)$

$$h(x) = \underbrace{5^x}_{f(x)} \underbrace{(x^4 - 3)}_{g(x)}$$

$$h'(x) = \underbrace{5^x \cdot \ln 5}_{f'} \cdot \underbrace{(x^4 - 3)}_{g} + \underbrace{5^x}_{f} \cdot \underbrace{4 \cdot x^3}_{g'}$$

$h(x) = f(x)/g(x)$ is differentiable ~~and~~ and

$$h'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}$$

$$h(x) = (e^x - 2)/(3 + 2x^5)$$

$$h'(x) = \frac{e^x(3 + 2x^5) - (e^x - 2)10x^4}{(3 + 2x^5)^2}$$

$h(x) = f(g(x))$ is differentiable and

$$h'(x) = g'(x) \cdot f'(g(x))$$

$h(x) = e^{3x+1} = f(g(x))$ with $g(x) = \underline{3x+1}$, $f(x) = e^x$

$$h'(x) = \underbrace{3}_{g'} \cdot \underbrace{e^{3x+1}}_{f'(g)}$$

$h(x) = \sqrt{5x+x^2+3} = f(g(x))$ with $g(x) = 5x+x^2+3$

$f(x) = \sqrt{x} = x^{1/2}$

$$h'(x) = \underbrace{(5+2x)}_{g'} \cdot \underbrace{\frac{1}{2\sqrt{5x+x^2+3}}}_{f'(g)}$$

$f'(x) = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

$h(x) = \ln(9+x^2)$ $\qquad h'(x) = 2x \cdot \frac{1}{9+x^2}$

OPTIMIZATION

$x_m \in A$ global min

Given $f : A \to \mathbb{R}$ ($A \subseteq \mathbb{R}$), $x_M \in A$ is said to be a max point for f if
global

$f(x_M) \geq f(x)$ for each $x \in A$, $x \neq x_M$

$f(x_m) \leq f(x)$ $x \neq x_m$

THEOREM (monotonicity test) Hp: $f : A \to \mathbb{R}$ (with $A \subseteq \mathbb{R}$) is differentiable in an interval (a, b) included in A

Ts: (i) $f'(x) > 0$ for each $x \in (a, b)$ implies f is monotone strictly increasing in (a, b)

(ii) $f'(x) < 0$ " " " implies that f is monotone strictly decreasing in (a, b)

(iii) $f'(x) = 0$ " " " implies that f is constant in (a, b)

$f : [-2, 3] \to \mathbb{R}, \ f(x) = \frac{3}{2}x^4 - 2x^3 - 6x^2 + 4$

$f'(x) = \frac{3}{2} \cdot 4 \cdot x^3 - 2 \cdot 3x^2 - 12x = 6x^3 - 6x^2 - 12x = 6x(x^2 - x - 2)$

x	-2 ... -1	-1 ... 0	0 ... 2	2 ... 3
$6x$	$- - -$	$- -$	$+ + + +$	$+ + +$
x^2-x-2	$+ + +$	$- -$	$- - - -$	$+ + +$
$f'(x)$	$- -$	$+ + +$	$- - - -$	$+ + +$

f is decreasing in $(-2, -1)$

f is increasing in $(-1, 0)$

f is decreasing in $(0, 2)$

f is increasing in $(2, 3)$

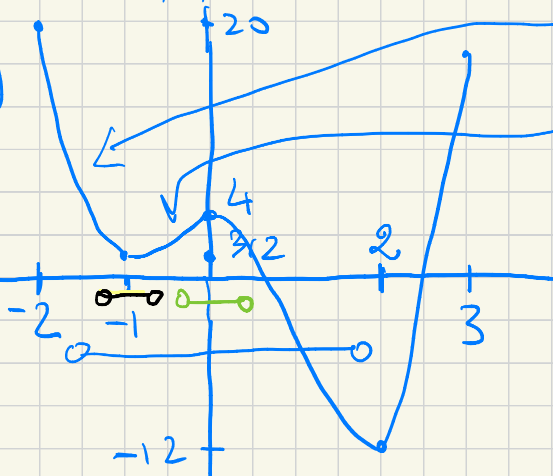
$f(-2) = 20, \ f(3) = 17.5$

$f(-1) = 3/2, \ f(0) = 4, \ f(2) =$

$x_M = -2, \ \max f = 20$

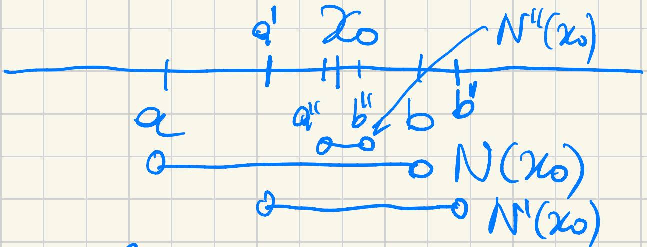
$x_m = 2, \ \min f = -12$

$R_f = [-12, 20]$



ANOTHER APROACH TO OPIMIZATION

DEF Given $x_0 \in \mathbb{R}$, a neighborhood of x_0, denoted $N(x_0)$, is any interval (a,b) such that $a < x_0 < b$



DEF Given a $f : A \to \mathbb{R}$ ($A \subseteq \mathbb{R}$), a point $x_0 \in A$ is said to be a <u>local max min point for f</u> if there exists at least one $N(x_0)$ such that

$$f(x_0) \geq f(x) \quad \leq \quad \text{for each } x \in N(x_0) \cap A,\ x \neq x_0$$

For the previous function, $x_0 = 0$ is a local max point: just take $N(0)$ equal to the green interval

$x_0 = -1$ is a local min point for f: use $N(-1)$ equal to the black interval

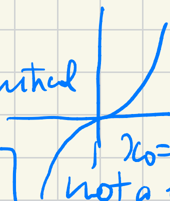
x_0 is a local max point min $\Leftarrow$ $\not\Rightarrow$ x_0 is a global max point min

THEOREM
(Fermat's theorem)

Hp: $f: A \to \mathbb{R}$ $(A \subseteq \mathbb{R})$, $x_0 \in A$
is a local max (min) point for f,
there exists at least one $N(x_0)$ included in A } x_0 is an interior point of A
f is differentiable at x_0

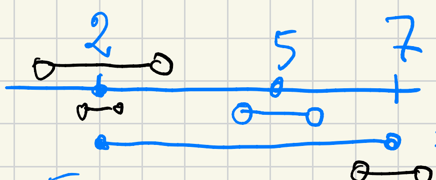
Ts: $f'(x_0) = 0$

$f(x) = x^3$
$f'(x) = 3x^2$
$x_0 = 0$ is a critical point



$x_0 = 0$ is not a local max/min point

global max (min) point + regularity $\Rightarrow$ x_0 is local max (min) point for f + regularity $\Longrightarrow$ $f'(x_0) = 0$
$\not\Longleftarrow$
x_0 is a critical point for f



$= [2,7] = A$

$x_0 = 5$ is interior for $[2,7]$
$x_0 = 2$ is interior for A? No $x_0 = 7$ is interior for A? No

For the previous function,
$f'(x) = 6x(x^2 - x - 2)$, it is $= 0$ at
$x_1 = 0$, $x_2 = 2$, $x_3 = -1$: critical points for f

x_0, indeed is a local max point
x_1 is a global min point
x_2 is a local min point.
But $x=-2$ is the global max point and is not a critical point because ~~the~~ it is not interior for $A=[-2,3]$, hence Fermat's theorem does not apply to $x=-2$.

In general, once a critical point x_0 is determined, it may be a local max point, or a local min point, or neither. One way to find out is to examine the sign of $f'(x)$ for x close to x_0. For instance, if $f'(x)>0$ for $x<x_0$, close to x_0, and $f'(x)<0$ for x close to x_0, $x>x_0$, then x_0 is a local max point

Another method relies on $f''(x_0)$.

DEF If $f:(a,b)\to\mathbb{R}$ is differentiable in (a,b), and moreover f' is differentiable in (a,b), then f is said to be twice differentiable in (a,b), and the derivative of f' is denoted by f''.

$f:\mathbb{R}^2\to\mathbb{R}$, $f(x)=7x^9+3x^4$, $f'(x)=7\cdot 9\cdot x^8+3\cdot 4x^3$

$f''(x)=7\cdot 9\cdot 8\cdot x^7+3\cdot 4\cdot 3\cdot x^2$

THEOREM. Hp: $f:A\to\mathbb{R}$ is twice differentiable in a neighborhood of x_0 and x_0 is a critical point for f

Ts: If $f''(x_0)<0$, then x_0 is a local max point for f

If $f''(x_0)>0$, " x_0 " " " min " " "

For the previous function, $x_1 = 0,\ x_2 = 2,\ x_3 = -1$ critical points

$f'(x) = 6x^3 - 6x^2 - 12x$, $f''(x) = 18x^2 - 12x - 12$

$f''(0) = -12 < 0$: $x_1 = 0$ is a local max point

$f''(2) = 18\cdot 4 - 12\cdot 2 - 12 = 72 - 24 - 12 = 36 > 0$

$x_2 = 2$ is a local min point

$f''(-1) = 18\cdot 1 - 12(-1) - 12 = 30 - 12 = 18 > 0$

$x_3 = -1$ is a local min point

it does not say whether x_2 or x_3 is a global min point

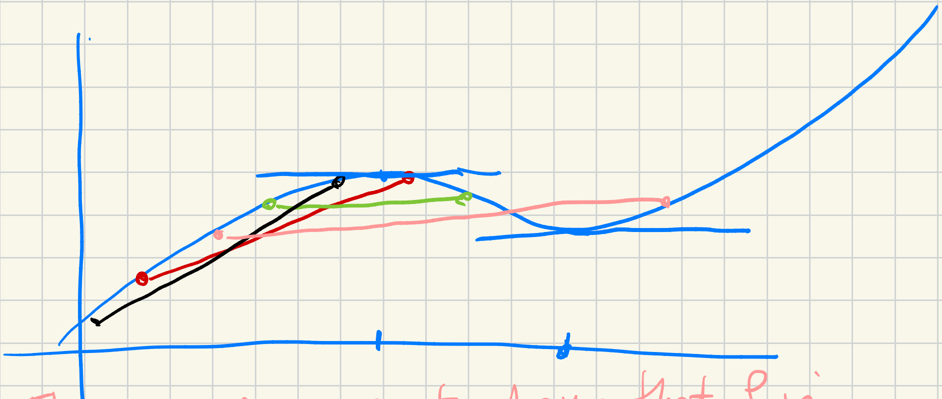
Theorem (Extreme Value theorem)

Hp: $f : [a,b] \to \mathbb{R}$ is continuous in $[a,b]$

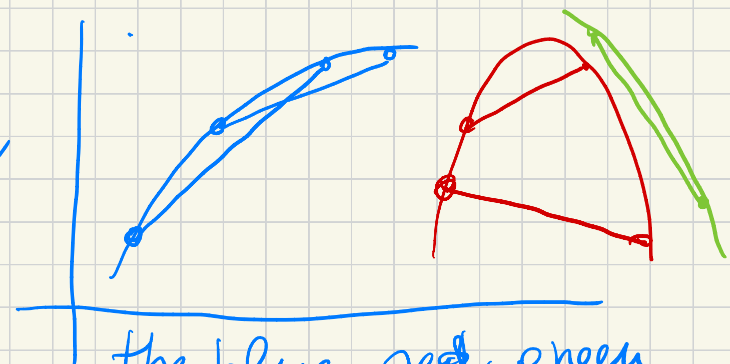
Ts: There exists at least one global max point for f, and at least one global minimum point for f

CONCAVE / CONVEX FUNCTIONS

DEF A function f defined over an interval A, that is $f : A \to \mathbb{R}$ (A is an interval) is said to be
convex
concave if for each two points on the graph, the segment joining them lies below the graph, or lies exactly on the
above
graph.



The pink segment shows that f is not concave nor convex



the blue, red, green graph is the graph of a concave function

CONCAVITY / CONVEXITY TEST

Suppose that $f:(a,b)\to\mathbb{R}$ is twice differentiable in (a,b)

If $f''(x)\le 0$ for each $x\in(a,b)$, then f is concave in (a,b)

If $f''(x)\ge 0$ " " " convex " "

$f(x)=\ln x-2x$, $f:(0,+\infty)\to\mathbb{R}$,

$\frac{1}{x}-2=0 \iff \frac{1}{x}=2 \Rightarrow x_0=1/2$

$x_0=1/2$ is a glob. max point for f

$f'(x)=\frac{1}{x}-2$, $f''(x)=-\frac{1}{x^2}<0$ for each $x\in(0,+\infty)$ hence f is concave

$g(x)=\frac{1}{2}x-4\sqrt{x+1}$ $g:(-1,+\infty)\to\mathbb{R}$

$(x+1)^{1/2}$

$\frac{1}{2}-\frac{2}{\sqrt{x+1}}=0 \iff \frac{1}{2}=\frac{2}{\sqrt{x+1}} \iff \frac{1}{4}=\frac{4}{x+1} \iff x+1=16 \iff x_0=15$

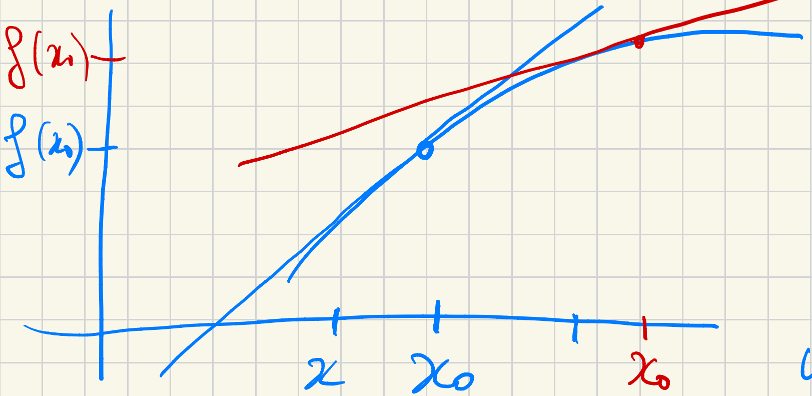
$x_0=15$ is a global min point for g

$g'(x)=\frac{1}{2}-4\cdot\frac{1}{2}(x+1)^{-1/2}=\frac{1}{2}-2(x+1)^{-1/2}$

$g''(x)=-2\left(-\frac{1}{2}\right)\cdot(x+1)^{-3/2}=\frac{1}{(x+1)^{3/2}}>0$ for each $x\in(-1,+\infty)$ hence g is convex

If $f: A \to \mathbb{R}$ is concave in A, then

$$\underbrace{f(x) \le f(x_0) + f'(x_0)(x - x_0)}_{\text{expression of the tangent line to the graph of } f \text{ at } (x_0, f(x_0))} \quad \text{for each } x \text{ and } x_0 \text{ in } A.$$



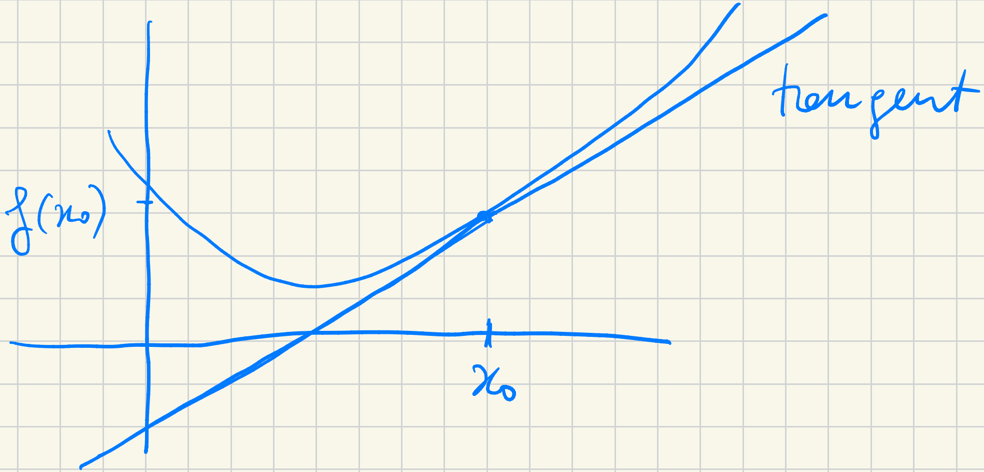
Suppose that f is concave and x_0 is a critical point, that is $f'(x_0) = 0$. Then the inequality is written as

$$f(x) \le f(x_0) \quad \text{for each } x \text{ in } A$$

Hence x_0 is global max point for f.

If the function to maximize is concave and x_0 is a critical point, then certainly x_0 is a global max point.

If f is convex, then the inequality above holds with $\ge$, and if x_0 is a critical point for f, then x_0 is a global min point for f.



tangent
$f(x_0)$
x_0